Solutions Manual

prepared by
E. M. Purcell
to accompany

electricity and magnetism

Berkeley Physics Course—Vol. II

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Solutions Manual

to accompany

ELECTRICITY AND MAGNETISM

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Berkeley Physics Course Volume II

Preface to the Solutions Manual

This manual of problem solutions is intended primarily as an aid to instructors teaching from the second volume of the Berkeley Physics Course. They will, I hope, find the manual a useful supplement to the text. The solutions for the less routine problems often include some explanation going beyond the bare requirements of the problem. The instructor would be well advised to peruse the solutions for the more ambitious problems before he assigns them—if only to be warned of traps which were not always foreseen when the problems were devised.

The solutions were checked by Thomas Knight, who found some errors and made many valuable suggestions for improvement and clarification. (Any instructor who feels that the morale in his class would be improved if he revealed that the author made some mistakes in working his own problems is hereby invited to do so.) Numerical errors may still turn up. Notice of such would be most welcome, as would any other suggestion for improvement of the material in the manual. Please address any suggestions to the undersigned at Lyman Laboratory, Harvard University, Cambridge, Mass. 02138.

Appropriate parts of this manual may be detached and posted, or with copyright notice displayed, may be reproduced in part in numbers for the students. We do not, however, recommend that the entire manual be made available to students.

E. M. Purcell
Cambridge, May 1966

CHAPTER 1

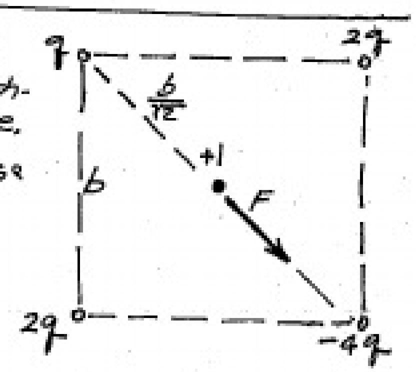
1.1 (a) Electrical repulsion: $F_e = e^2/r^2$
Gravitational attraction: $F_g = Gm^2/r^2$

$$\frac{F_e}{F_g} = \frac{e^2}{Gm^2} = \frac{(4.8\times10^{-10})^2}{6.67\times10^{-8}\times(.911\times10^{-27})^2} = 4.15\times10^{42}$$

To make $F_e/F_g = 1$, the electron mass would have to be increased by the factor $(4.15\times10^{42})^{\frac{1}{2}}$, becoming 1.86×10^{-6} gm.

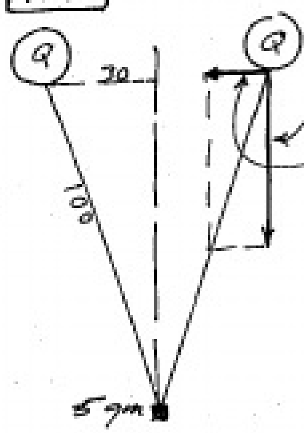
(b) If unit mass were defined by specifying that unit masses one cm apart attract each other with a force of one dyne, the unit would be equivalent to a mass in grams, m, such that $Gm^2 = 1$, or $m = 1/\sqrt{G}$ $= 3900$ gm. A man's mass might be 20 such units. On the other hand, the dyne is itself derived from the gram, centimeter, and second. One might more logically, perhaps, define new units of both mass and force by requiring that Newton's second law be written $F = ma$ and the gravitational law $F = m^2/r^2$, with a in cm/sec^2, r in cm. The unit of mass thus defined is equivalent to 1.5×10^7 gm while the unit of force is equivalent to 1.5×10^7 dynes. (If one proceeds to define a new unit of electric charge for this system, so that Coulomb's law retains the appearance $F = q^2/r^2$, a unit equivalent to 3900 esu is called for.)

1.2 The forces arising from the charges in the northeast and southwest corners are equal and opposite. Resultant force on the positive charge is southeast, with magnitude:



$$F = \frac{4q}{\left(\frac{b}{\sqrt{2}}\right)^2} + \frac{1q}{\left(\frac{b}{\sqrt{2}}\right)^2} = \frac{10q}{b^2}$$

1.3



Vertical component of force exerted by thread on balloon is $\frac{1}{2} \times 5 \times 980 = 2450$ dynes

Horizontal component

$= 2450 \frac{30}{\sqrt{100^2 - 30^2}} = 2450 \frac{30}{95.5}$

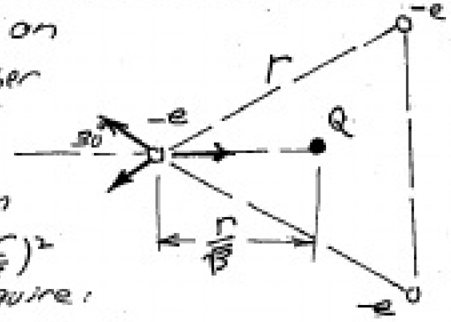
$= 770$ dynes, to be balanced by Coulomb repulsion:

$770 = \frac{Q^2}{60^2}$, which gives:

$Q = 1660$ esu

1.4

The resultant force on one electron due to the other two is $2 \frac{e^2}{r^2} \cos 30° = \sqrt{3} \frac{e^2}{r^2}$



The opposing force arising from the central charge is $eQ/(\frac{r}{\sqrt{3}})^2$

For total force zero we require:

$$\frac{3eQ}{r^2} = \frac{\sqrt{3}\, e^2}{r^2} \quad \text{or} \quad Q = e/\sqrt{3}$$

The equilibrium is unstable. If, with all the other charges held fixed, one electron were displaced outward, the inward force on it from Q would decrease more than the outward force from the other two electrons. Thus a net outward force force would arise, tending to increase the displacement.

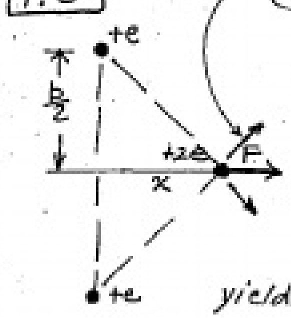
1.5

(a) The force on the positive charge is zero. The force on one of the negative charges is $\frac{e^2}{a^2} - \frac{e^2}{(2a)^2} = \frac{3}{4}\frac{e^2}{a^2}$, directed toward the center.

(b) Potential energy $= -\frac{2e^2}{a} + \frac{e^2}{2a} = -\frac{3}{2}\frac{e^2}{a}$

The minus sign indicates that energy would have to be supplied to take the system apart; the potential energy increases as the distance a is made larger.

1.6



Force $= \frac{2e^2}{x^2 + (\frac{b}{2})^2}$

Resultant force $F = 2\left[\frac{2e^2}{x^2+(\frac{b}{2})^2}\right]\frac{x}{\sqrt{x^2+(\frac{b}{2})^2}}$

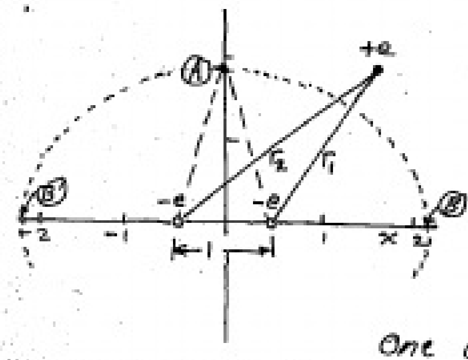
or $F = \text{constant} \times \frac{x}{(b^2+4x^2)^{3/2}}$

To find the maximum, set $\frac{dF}{dx} = 0$, yielding: $(b^2+4x^2)^{\frac{3}{2}} - 12x^2(b^2+4x^2)^{\frac{1}{2}} = 0$

or $b^2 - 8x^2 = 0$, from which: $x = \pm b/2\sqrt{2}$

1.7

Without loss of generality we can put the two electrons on the x axis unit distance apart. For any location of the proton the potential energy of the system is:



$U = \frac{e^2}{1} - \frac{e^2}{r_1} - \frac{e^2}{r_2}$. Thus for $U = 0$ we require:

$$\frac{1}{r_1} + \frac{1}{r_2} = 1 \quad (1)$$

One obvious location meeting the requirement is Ⓐ, with $r_1 = r_2 = 2$. To locate a position on the positive x axis, with $r_1 = x - \frac{1}{2}$, $r_2 = x + \frac{1}{2}$: $\frac{1}{x-\frac{1}{2}} + \frac{1}{x+\frac{1}{2}} - 1 = 0$. This gives the quadratic equation $2x - x^2 + \frac{1}{4} = 0$ with roots $x = 2.118$ and $x = -.118$. The latter root must be thrown out, for with $x < \frac{1}{2}$, r_1 and r_2 are not represented by $x - \frac{1}{2}$ and $x + \frac{1}{2}$. We are left with the point Ⓑ at $x = 2.118$. Of course B', at $x = -2.118$ is equally good. Eq. 1 defines a curve in the x-y plane, and a surface of revolution around the x-axis, in space. This surface is the set of all points where the proton can be placed to give $U = 0$. The surface looks something like a prolate ellipsoid, but it isn't.

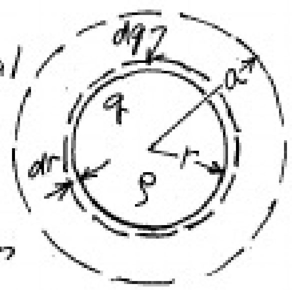
1.8 

Suppose the array has been built from the left as far as the negative ion A. To add the next ion B, a positive ion, requires an amount of work:

$$-\frac{e^2}{s} + \frac{e^2}{2s} - \frac{e^2}{3s} + \cdots = -\frac{e^2}{s}\left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right]$$

The expansion of $\ln(1+x)$ is $1 - \frac{x}{2} + \frac{x^2}{3} - \cdots$, converging for $-1 < x \leq 1$. Evidently the sum in brackets above is just $\ln 2$, or 0.693. Thus the energy of the chain, per ion, is $-0.693\frac{e^2}{s}$

1.9 The charge in the sphere of radius r is: $q = \frac{4\pi}{3}r^3\rho$. Its external field is the same as if q were at the center. The next shell to be added contains charge dq: $dq = 4\pi r^2 dr\,\rho$. The work done in bringing up dq is therefore:



$$dW = \frac{q \cdot dq}{r} = \left(\frac{4\pi\rho}{3}\right)^2 r^4\,dr.$$

Building up the whole sphere this way requires the work:

$$W = \int_0^a \left(\frac{4\pi\rho}{3}\right)^2 r^4\,dr = \left(\frac{4\pi\rho}{3}\right)^2 \frac{a^5}{5}$$

The charge in the complete sphere is: $Q = \frac{4\pi}{3}a^3\rho$. Thus the potential energy U, which is the same as W, can be written: $U = \frac{3}{5}Q^2/a$. Notice that Q^2/a has the proper dimensions (charge)^2/distance. Indeed we could have predicted that much of our result without calculation.

1.10 Setting the potential energy $\frac{3}{5}\frac{e^2}{r_0}$ equal to mc^2 and solving for r_0:

$$r_0 = \frac{3}{5}\frac{e^2}{mc^2} = \frac{3}{5}\frac{(4.80\times10^{-10})^2}{(.91\times10^{-27})(3\times10^{10})^2} = 1.69\times10^{-13}\text{ cm}$$

1.11 ρ_0 $Q = 92e$, radius a, $U = \frac{3}{5}\frac{Q^2}{a}$

ρ_0 $Q' = \frac{Q}{2}$ ρ_0 $Q' = \frac{Q}{2}$, radius $a' = \frac{a}{2^{1/3}}$, $U' = \frac{3}{5}\frac{Q'^2}{a'}$

$$U' = \frac{3}{5}\frac{(Q/2)^2}{(a/2^{1/3})} = \frac{3}{5}\frac{Q^2}{a}\cdot\frac{1}{2^{5/3}}$$

The decrease in energy is $U - 2U'$

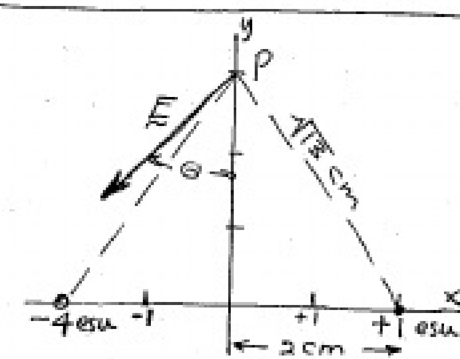
$$U - 2U' = \frac{3}{5}\frac{Q^2}{a}\left[1 - \frac{2}{2^{5/3}}\right] = \frac{3}{5}\frac{Q^2}{a}(.370) = .222\frac{Q^2}{a}$$

With ρ_0 given, a is determined by $\frac{4\pi}{3}a^3\rho_0 = Q$, from which $Q^2/a = \left(\frac{4\pi\rho_0}{3}\right)^{1/3}Q^{5/3}$ Thus:

$$U - 2U' = .222\left[\frac{4\pi}{3}\cdot 4\times10^{28}\right]^{\frac{1}{3}}\left[92\times4.8\times10^{-10}\right]^{5/3} = 6.7\times10^{-4}\text{ ergs}$$

1 MeV $= 1.6\times10^{-6}$ ergs. Hence $U - 2U' = 420$ MeV.

1.12 (a) At the point P,



$$E_x = -\frac{1}{(\sqrt{13})^2}\cdot\frac{2}{\sqrt{13}} - \frac{4}{(\sqrt{13})^2}\cdot\frac{2}{\sqrt{13}} = -\frac{10}{13^{3/2}}\text{ statvolts/cm}$$

$$E_y = \frac{1}{(\sqrt{13})^2}\cdot\frac{3}{\sqrt{13}} - \frac{4}{(\sqrt{13})^2}\cdot\frac{3}{\sqrt{13}} = \frac{-9}{13^{3/2}}\text{ statvolts/cm}$$

$$E = (E_x^2 + E_y^2)^{\frac{1}{2}} = \left(\frac{100+81}{13^3}\right)^{\frac{1}{2}} = (.0825)^{\frac{1}{2}} = 0.288$$

$E = 0.288$ statvolts/cm. $\theta = \tan^{-1}\frac{10}{9} = 48°$

(b) A point where the field is zero must lie on the x-axis, nearer the smaller of the two charges, but not between them. The condition is $\frac{1}{(x-2)^2} = \frac{4}{(x+2)^2}$. This has two roots $x = 6$ and $x = \frac{2}{3}$. At $x = \frac{2}{3}$ the fields of the two sources are equal but not opposite. The only point where the field is zero is the point $x = 6$, $y = 0$, $z = 0$.

1.13 At the surface of a drop of radius r the field strength is Q/r^2. With $r = 0.005$ cm and $E = 20$ statvolts/cm, $Q = Er^2 = 5\times10^{-4}$ esu. The drop weighs $\frac{4\pi}{3}(0.005)^3(1.00)\,980$ dynes, or 5×10^{-4} dynes. A vertical field E_v will support the drop if $QE_v = mg$, or $E_v = \frac{mg}{Q} = \frac{5\times10^{-4}\text{ dynes}}{5\times10^{-4}\text{ esu}} = 1.0$ statvolt/cm

1.14 $\sigma = \frac{E}{4\pi} = \frac{10^{-3}}{4\pi}\frac{\text{esu}}{\text{cm}^2}$. The equivalent number of electrons per cm² is $\sigma/e = \frac{10^{-3}}{4\pi\times4.8\times10^{-10}} = 1.6\times10^{5}$

1.15 The total amount of negative charge is $\int_0^\infty 4\pi r^2 dr\cdot\rho$. Given that ρ has the form $C\epsilon^{-2r/a_0}$ we can easily determine C so as to make the distribution contain one electronic charge: $\int_0^\infty 4\pi r^2 C\,\epsilon^{-2r/a_0}dr = e$ (We use ϵ for 2.718··· so as to avoid confusion with the electron charge e.) Substituting $x = 2r/a_0$ we have

$$\frac{4\pi C}{e}\left(\frac{a_0}{2}\right)^3\int_0^\infty x^2\epsilon^{-x}dx = 1 \qquad \text{Since } \int_0^\infty x^2\epsilon^{-x}dx = 2,$$

we get: $C = \frac{e}{\pi a_0^3}$. To calculate the amount of charge inside $r = a_0$, we must integrate only as far as $r = a_0$, or $x = 2$:

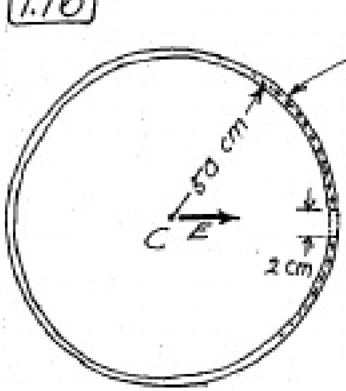
$$-q = 4\pi C\int_0^{a_0} r^2\epsilon^{-2r/a_0}dr = \frac{e}{2}\int_0^2 x^2\epsilon^{-x}dx$$

$$= \frac{e}{2}\left[-(x^2+2x+2)\epsilon^{-x}\right]_0^2 = e\left(1-\frac{5}{\epsilon^2}\right) = 0.323e$$

At the radius $r = a_0$, then, the field is that of the central charge $+e$ surrounded by a negative charge $-0.323e$, leaving a net positive charge of $0.677e$.

$$E_r = \frac{0.677e}{a_0^2} = \frac{0.677\times4.8\times10^{-10}}{(0.53\times10^{-8})^2} = 1.15\times10^{7}\text{ statvolts/cm}$$

1.16

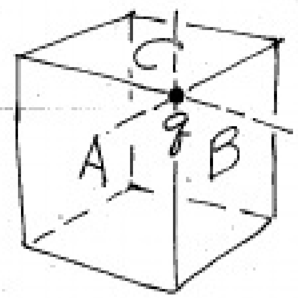


To make a complete, uniformly charged ring, we would need to add a 2-cm long rod carrying a charge $\frac{2}{312}$ esu. The field at C would then be zero. Hence, with the gap unfilled the field at C must be that of a negative charge of magnitude $\frac{2}{312}$ esu distributed over the gap. Because 2 cm ≪ 50 cm, we may treat this distribution as a point charge in calculating the field it would produce at C:

$$E = \left(\frac{2}{312}\text{ esu}\right)\frac{1}{(50\text{ cm})^2} = 2.57\times10^{-6}\text{ statvolts/cm}$$

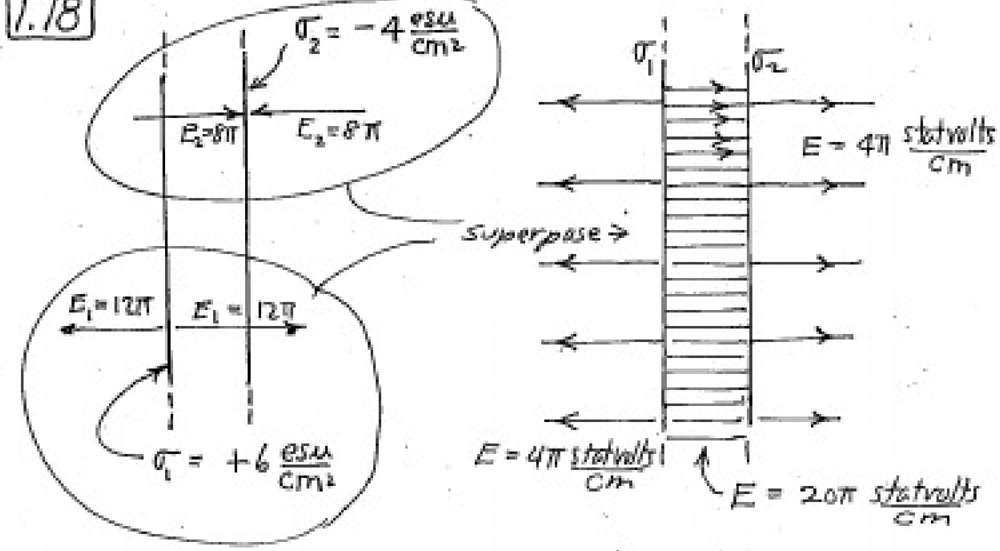
1.17 (a) The total flux is $4\pi q$ (Gauss's law). The flux through every face of the cube must be the same, because of symmetry. Hence over any one of the six faces: $\int \underset{\sim}{E}\cdot d\underset{\sim}{a} = \frac{4\pi q}{6} = \frac{2\pi}{3}q$

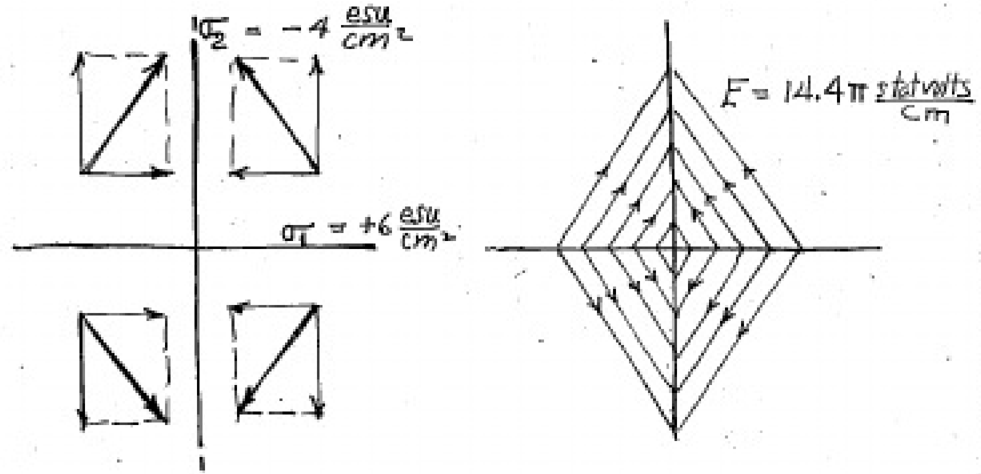
(b)



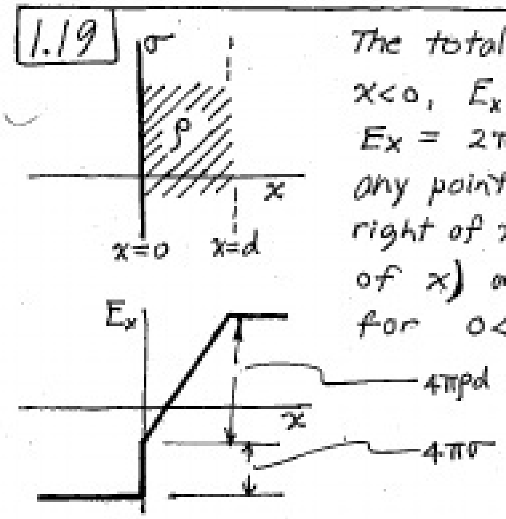
Because the field of q is parallel to the surface on each of the three faces A, B and C, the flux through these faces is zero. The flux through the other three faces must therefore add up to $4\pi q/8$, because our cube is one of eight such surrounding q. The three faces being symmetrically located with respect to q, the flux through each must be $\frac{1}{3}\cdot\frac{4\pi q}{8} = \frac{\pi q}{6}$

1.18





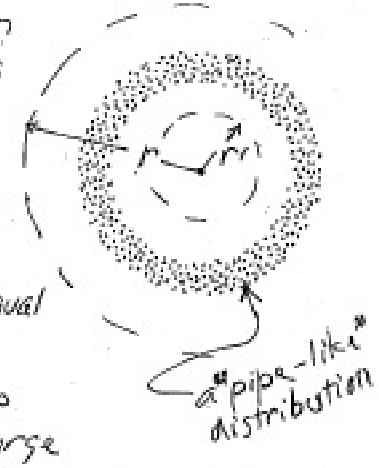
1.19



The total charge per cm² is $\sigma + \rho d$. For $x<0$, $E_x = -2\pi(\sigma+\rho d)$ and for $x>d$ $E_x = 2\pi(\sigma+\rho d)$. More generally, at any point x, $E_x = -2\pi$ (charge per cm² to right of x) $+ 2\pi$ (charge per cm² to left of x) or $E_x = -2\pi(d-x)\rho + 2\pi(\sigma + x\rho)$ for $0<x<d$. $E_x = 2\pi\sigma - 2\pi\rho d + 4\pi\rho x$

1.21

If the charge distribution has axial symmetry, so must its field. $\underset{\sim}{E}$ must be radial, and can depend on r only. The flux through a cylinder of radius r and length 1 cm is $2\pi r E_r$ and by Gauss's law this must equal $4\pi\lambda$, where λ is the charge enclosed, per unit axial length. So $E_r = \frac{2\lambda}{r}$ just as if the charge were concentrated on the axis. Inside the hollow distribution the charge enclosed by a cylinder of radius r' is zero. Hence $E_r = 0$ and $\underset{\sim}{E} = 0$ everywhere inside the hollow cylindrically symmetrical charge distribution.

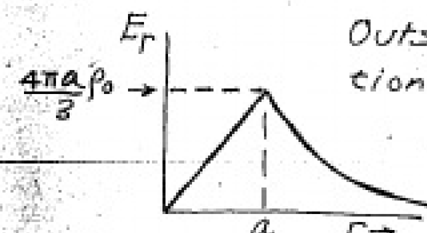


1.22



On the sphere of radius r, the field is the same as if all charge inside that sphere, which amounts to $\frac{4\pi r^3 \rho_0}{3}$, were at the center:

$$E_r = \left(\tfrac{4}{3}\pi r^3 \rho_0\right)/r^2 = \frac{4\pi\rho_0 r}{3} \qquad (r<a)$$

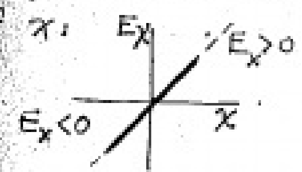


Outside the whole charge distribution the field is

$$E_r = \frac{4}{3}\frac{\pi a^3 \rho_0}{r^2} \qquad (r>a)$$

E_r is <u>continuous</u> at $r=a$

To discuss the continuity of the electric field in the neighborhood of the origin, look at E_x as a function of x:

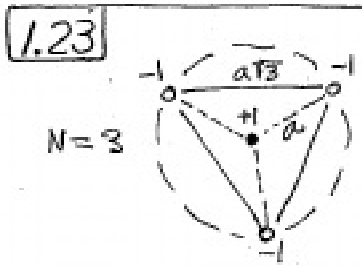


Clearly there is no discontinuity; there is no singularity whatever. The singularity suggested by plotting E_r like this:



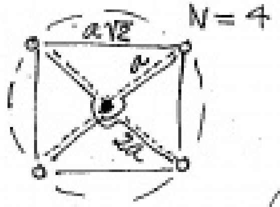
is spurious. There is, properly, no negative range of r.

1.23

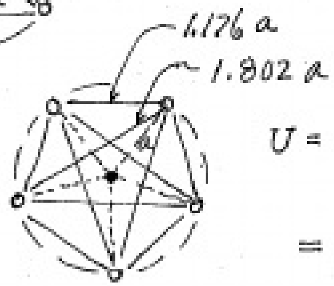


$$U = \sum_{\text{all pairs}} \frac{q_j q_k}{r_{jk}}$$

$$U = -\frac{3}{a} + \frac{3}{\sqrt{3}a} = -\frac{1.268}{a}$$

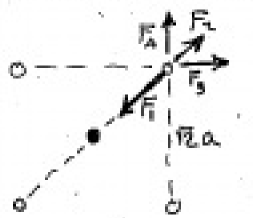


$$U = -\frac{4}{a} + \frac{4}{\sqrt{2}a} + \frac{2}{2a} = -\frac{.172}{a}$$



$$U = -\frac{5}{a} + \frac{5}{1.176a} + \frac{5}{1.802a}$$

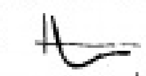
$$= \frac{5}{a}(-1 + .850 + .555) = +\frac{2.03}{a}$$



To find the force on one of the negative charges in the case N = 4: Resultant outward force

$$= -F_1 + F_2 + 2F_3 \cos 45°$$

$$= -\frac{1}{a^2} + \frac{1}{(2a)^2} + 2\left(\frac{1}{2a^2}\right)\frac{1}{\sqrt{2}} = -\frac{.043}{a^2}$$

Thus force is <u>inward</u>. This is consistent with the negative U for N = 4. Negative U insures that the particles will fly together, if released, because the potential energy is then lowered indefinitely by shrinking a. Positive U, as in N = 5, implies that the energy is lowered by increasing a. [An argument like this might not work if the potential energy had a more complicated dependence on the interparticle distance — for instance, if the potential energy of two particles varied like this:  as a function of the distance between them.]

1.24 All the charge is on the surface of the sphere. The potential energy, the work required to assemble the charge distribution, is:

$$U = \int_0^Q \frac{q\,dq}{R} = \frac{1}{2}\frac{Q^2}{R},$$

since, with charge q already in place, the work done in bringing up the next element of charge dq is $(q/R)\,dq$. [Note how this differs from the potential energy of the sphere with a uniform volume density of charge, considered in Problem 1.9. In that case, $U = \frac{3}{5}Q^2/R$.]

After division into two drops of equal radius R':

$R' = R/2^{1/3}$ (total volume constant) and $Q' = Q/2$. For one of these drops, then, $U' = \frac{1}{2}Q'^2/R' = \frac{1}{2}\left(\frac{Q}{2}\right)^2\left(\frac{2^{1/3}}{R}\right) = \frac{1}{2^{8/3}}\frac{Q^2}{R}$. The <u>decrease</u> in electrical energy is therefore:

$$U - 2U' = \frac{Q^2}{2R}\left(1 - \frac{1}{2^{2/3}}\right) = \frac{0.185\,Q^2}{R}$$

The increase in surface area is $2\cdot 4\pi R'^2 - 4\pi R^2 = 3.27R^2$. Thus the increase in surface energy (work done in breaking intermolecular bonds) is

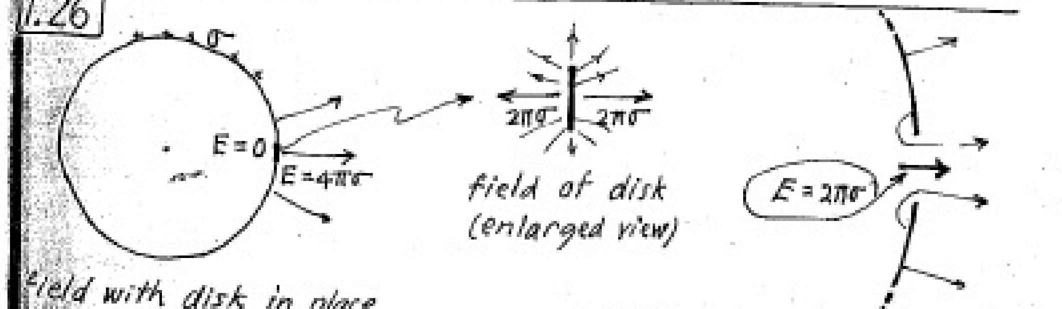
$$50 \text{ ergs/cm}^2 \times 3.27R^2 = 163\,R^2 \text{ ergs}$$

Putting this equal to the decrease in U:

$$163\,R^2 = \frac{0.185\,Q^2}{R} \quad \text{or} \quad Q^2 = \frac{163}{.185}R^3,$$

with Q in esu and R in cm. With R given as 10^{-2} cm, $Q = (8.8\times10^{-4})^{1/2} = 2.97\times10^{-2}$ esu. At the surface the electric field strength is $Q/R^2 = 297$ statvolts/cm.

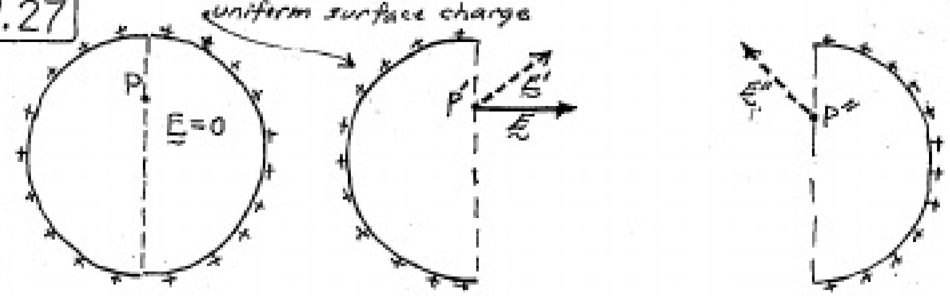
1.26



Field with disk in place

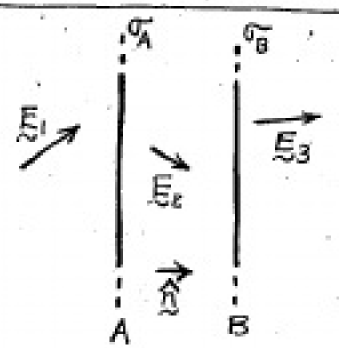
Field at center of hole must be 2πσ outward.

1.27



Consider the complete spherical shell formed by joining two such bowls. The field inside must be zero. But if the field at P' of the isolated hemisphere had a <u>radial</u> component, as does $\underset{\sim}{E}'$ above, the other hemisphere, at the corresponding point P'', would have to have a radial component of the same sign. That is, $\underset{\sim}{E}'$ implies, by symmetry, $\underset{\sim}{E}''$. These cannot give $\underset{\sim}{E}=0$ when superposed. Hence $\underset{\sim}{E}$ must be perpendicular to the plane spanning the hemisphere.

1.29



Force on A, per cm²: $\sigma_A \dfrac{(\underset{\sim}{E}_1+\underset{\sim}{E}_2)}{2}$

Force on B, per cm²: $\sigma_B \dfrac{(\underset{\sim}{E}_2+\underset{\sim}{E}_3)}{2}$

From Gauss's law:

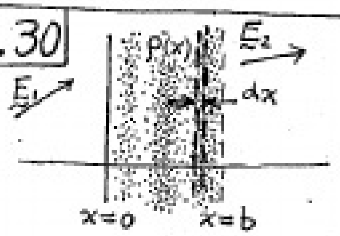
$\underset{\sim}{E}_2-\underset{\sim}{E}_1 = 4\pi\sigma_A\,\hat{\underset{\sim}{n}}$
$\underset{\sim}{E}_3-\underset{\sim}{E}_2 = 4\pi\sigma_B\,\hat{\underset{\sim}{n}}$
} use these to eliminate E_2 as follows:

Total force on both sheets $= \frac{\sigma_A}{2}(\underset{\sim}{E}_1+\underset{\sim}{E}_2) + \frac{\sigma_B}{2}(\underset{\sim}{E}_2+\underset{\sim}{E}_3)$

$$= \frac{\sigma_A}{2}(\underset{\sim}{E}_1+\underset{\sim}{E}_3-4\pi\sigma_B\hat{\underset{\sim}{n}}) + \frac{\sigma_B}{2}(\underset{\sim}{E}_3+\underset{\sim}{E}_1+4\pi\sigma_A\hat{\underset{\sim}{n}})$$

$$= (\sigma_A+\sigma_B)\frac{(\underset{\sim}{E}_1+\underset{\sim}{E}_3)}{2}$$

1.30



Let σ be the total charge between $x=0$ and $x=b$, per cm² of slab: $\sigma = \int_0^b \rho\, dx$. The x component of force on the slab, per cm² is

given by: $F_x = \int_0^b \rho E_x\, dx$

Applying Gauss's law to a thin box of area 1 cm², thickness dx, $E_x(x+dx) - E_x(x) = 4\pi\rho\, dx$, since $\rho\, dx$ is the charge in the box. From this,

$\dfrac{dE_x}{dx} = 4\pi\rho$ (a special case of the divergence relation we'll meet in Chapter 2). Substituting for ρ:

$$F_x = \int_0^b \frac{1}{4\pi} E_x \frac{dE_x}{dx} = \frac{1}{8\pi}\int_0^b \frac{d}{dx}E_x^2\, dx = \frac{1}{8\pi}\left(E_{2x}^2 - E_{1x}^2\right)$$

This can be written $F_x = \left(\dfrac{E_{2x}-E_{1x}}{4\pi}\right)\left(\dfrac{E_{2x}+E_{1x}}{2}\right)$

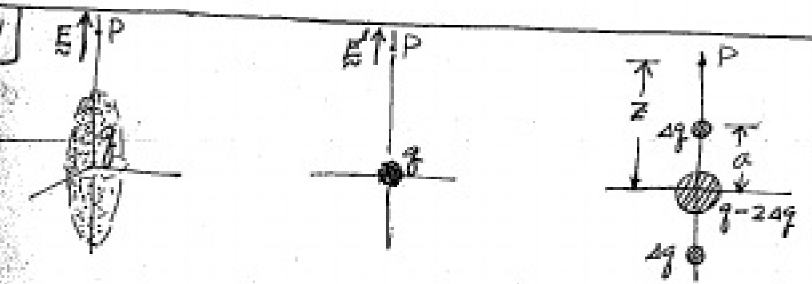
The first factor is nothing but σ, since:

$$\sigma = \int_0^b \rho\, dx = \int_0^b \frac{1}{4\pi}\frac{dE_x}{dx}\, dx = \frac{E_{2x}-E_{1x}}{4\pi}$$

Thus $F_x = \sigma\left(\dfrac{E_{2x}+E_{1x}}{2}\right)$

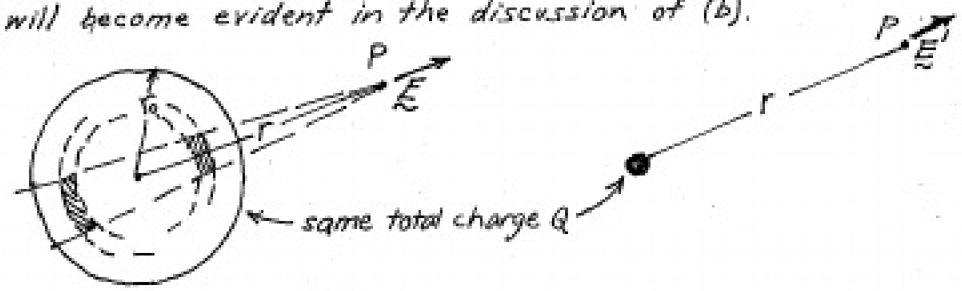
If $\underset{\sim}{E}$ has uniform y and z components arising from other sources, these are constant throughout the slab, producing force components F_y and F_z, per cm² of slab, given simply by σE_y and σE_z. So we can state more generally, $F = \sigma\dfrac{\underset{\sim}{E}_1+\underset{\sim}{E}_2}{2}$

1.31



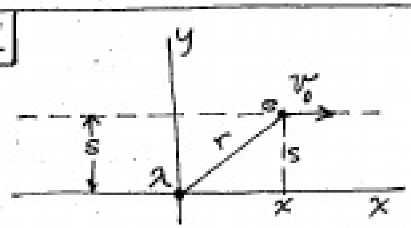
is greater than E'. To see that, think of the cigar-shaped distribution as being roughly equivalent to a central charge and two point charges at $z=\pm a$. Moving these charges, called Δq away from the center alters their contribution to field at P, from $\dfrac{2\Delta q}{z^2}$ to $\dfrac{\Delta q}{(z-a)^2}+\dfrac{\Delta q}{(z+a)^2}$. Clearly $\dfrac{1}{(z-a)^2}+\dfrac{1}{(z+a)^2} > \dfrac{2}{z^2}$.

1.32 (a) It would not be true for an inverse-cube law. This will become evident in the discussion of (b).



(b) If the force law is inverse-square, the "near" and the "far" parts of the spherical distribution make equal contribution to the field E at P. That is, the greater amount of charge in the portion ▧, as compared with ▧ is just off-set by its greater distance. It follows that for an inverse-cube force the nearer regions would dominate. If so, shrinking the distribution down to a central point, which would move the nearer portions of the source farther away from P, would weaken the field at P. We conclude: $E' < E$.

1.34



$$E = \frac{2\lambda}{r} \qquad E_y = \frac{2\lambda}{r}\cdot\frac{s}{r}$$

$$F_y = eE_y$$

$$dp_y = F_y\, dt = F_y \frac{dx}{v_0}$$

Treating the trajectory as a straight line $y = s$ traversed at the constant speed v_0, the transverse momentum acquired during the traversal is

$$p_y = \int_{-\infty}^{\infty} \frac{F_y\, dx}{v_0} = \frac{2\lambda e}{v_0}\int_{-\infty}^{\infty}\frac{s\, dx}{s^2 + x^2} = \frac{2\lambda e}{v_0}\left[\tan^{-1}\frac{x}{s}\right]_{-\infty}^{\infty} = \frac{2\pi\lambda e}{v_0}$$

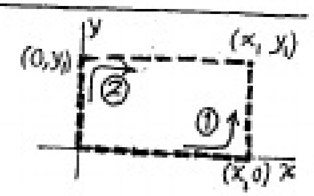
$p_x = mv_0$ Thus the angular deflection α is:

$$\alpha = \frac{p_y}{p_x} = \frac{2\pi\lambda e}{m v_0^2} = \frac{\pi\lambda e}{\text{electron kinetic energy}}$$

Note that λe has the dimensions of energy, being (esu/cm) esu. Our approximation is good if $\alpha \ll 1$.

CHAPTER 2

2.1



Path ①: $\displaystyle\int_{(0,0)}^{(x_1,y_1)} \mathbf{E}\cdot d\mathbf{s} = \int_0^{x_1} E_x(x,0)\,dx + \int_0^{y_1} E_y(x_1,y)\,dy$

$$= 0 + \int_0^{y_1}(3x_1^2 - 3y^2)\,dy = 3x_1^2 y_1 - y_1^3$$

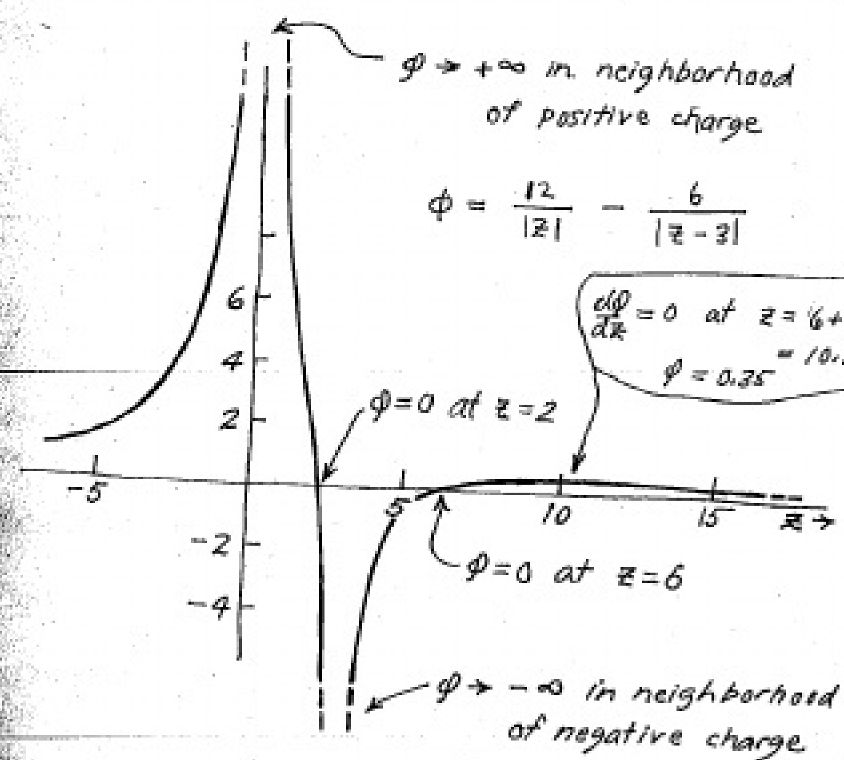
Path ②: $\displaystyle\int_{0,0}^{x_1,y_1} \mathbf{E}\cdot d\mathbf{s} = \int_0^{y_1} E_y(0,y)\,dy + \int_0^{x_1} E_x(x,y_1)\,dx$

$$= \int_0^{y_1} -3y^2\,dy + \int_0^{x_1} 6xy_1\,dx = -y_1^3 + 3x_1^2 y_1$$

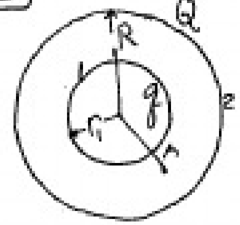
(same result)

The electric potential ϕ, if taken as zero at (0,0), is just the negative of this, since we define ϕ by $-\int \mathbf{E}\cdot d\mathbf{s}$, or $\mathbf{E} = -\nabla\phi$. That is, $\phi = y^3 - 3x^2 y$

2.2



2.3



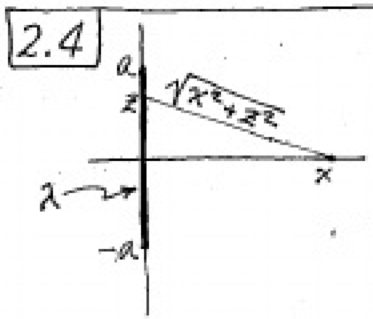
The charges will be uniformly distributed over their respective spheres, in this spherically symmetrical system. Let's relabel the radius of the inner sphere as r_1, leaving r for the general radial coordinate. Inside the larger sphere the electric field is that of q alone. The potential difference ϕ_{12} (potential of the inner sphere relative to the outer) is:

$$\phi_{12} = \int_2^1 -\mathbf{E}\cdot d\mathbf{s}$$

$$\mathbf{E} = \frac{q\hat{\mathbf{r}}}{r^2} \qquad d\mathbf{s} = \hat{\mathbf{r}}\,dr \qquad \phi_{12} = \int_R^{r_1} -\frac{q\,dr}{r^2} = \int_{r_1}^{R} \frac{q\,dr}{r^2}$$

$$= q\left[-\frac{1}{r}\right]_{r_1}^{R} = q\left(\frac{1}{r_1} - \frac{1}{R}\right) = \phi_1 - \phi_2$$

2.4



$$\phi(x) = \int \frac{dq}{r} = \int_{-a}^{a} \frac{\lambda\,dz}{\sqrt{x^2+z^2}}$$

$$= \lambda\left[\ln\left(z + \sqrt{x^2+z^2}\right)\right]_{z=-a}^{a}$$

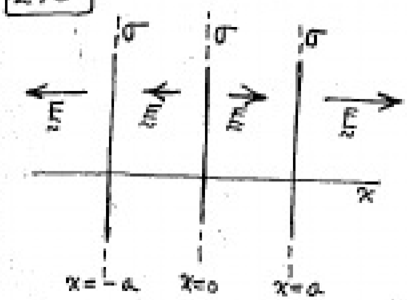
$$\phi(x) = \lambda \ln \frac{\sqrt{a^2+x^2}+a}{\sqrt{a^2+x^2}-a}$$

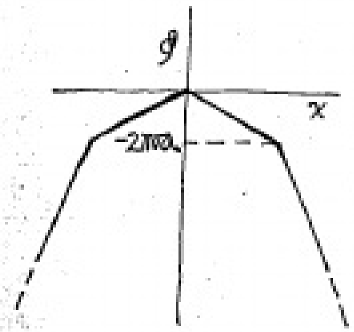
[As a check, observe that for $x \gg a$ this can be approximated as follows:

$$\ln \frac{\sqrt{a^2+x^2}+a}{\sqrt{a^2+x^2}-a} \approx \ln \frac{1+a/x}{1-a/x} \approx \frac{2a}{x}$$ Hence for

$x \gg a$, $\quad \phi(x) \approx \dfrac{2\lambda a}{x} = \dfrac{q}{x}$ where q is the total charge, $2\lambda a$, on the rod.]

2.5





$E_x = 2\pi\sigma, \qquad 0 < x < a$

$E_x = 6\pi\sigma, \qquad a < x$

$E_x = -2\pi\sigma \qquad -a < x < 0$

$E_x = -6\pi\sigma \qquad x < -a$

$$\phi = \int_0^x -E_x\,dx$$

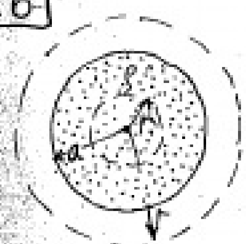
$\phi = -2\pi\sigma x \qquad 0 \le x \le a$

$\phi = 4\pi\sigma a - 6\pi\sigma x \qquad a \le x$

$\phi = 2\pi\sigma x \qquad -a \le x \le 0$

$\phi = 4\pi\sigma a + 6\pi\sigma x \qquad x \le -a.$

2.6



(a) Consider a cylinder of unit length, of radius $r < a$. Charge contained is $\pi r^2 \rho$. Area of surface is $2\pi r$; flux through surface is $2\pi r E$. Gauss's law says:

$$\underbrace{2\pi r E}_{\text{flux}} = 4\pi \underbrace{(\pi r^2 \rho)}_{\text{charge}}, \text{ from which, } E = 2\pi\rho r.$$

Considering a cylinder of radius $r > a$, which contains an amount of charge $\pi a^2 \rho$, we find

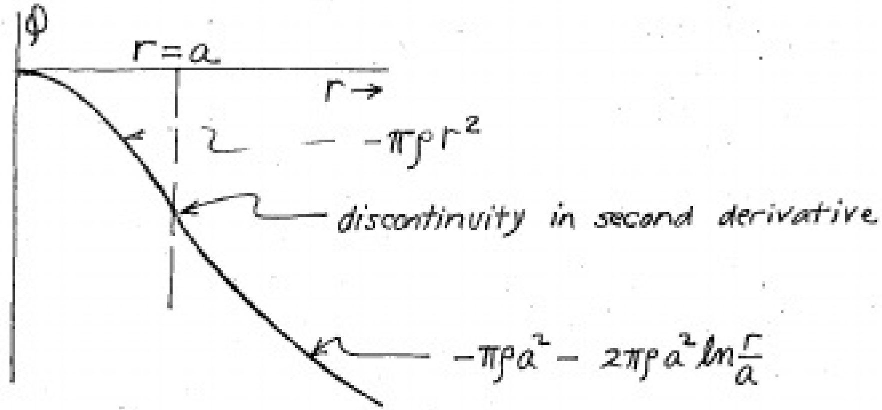
$$2\pi r E = 4\pi(\pi a^2 \rho), \text{ or } E = \frac{2\pi\rho a^2}{r}$$

(b) Take $\phi = 0$ at $r = 0$:

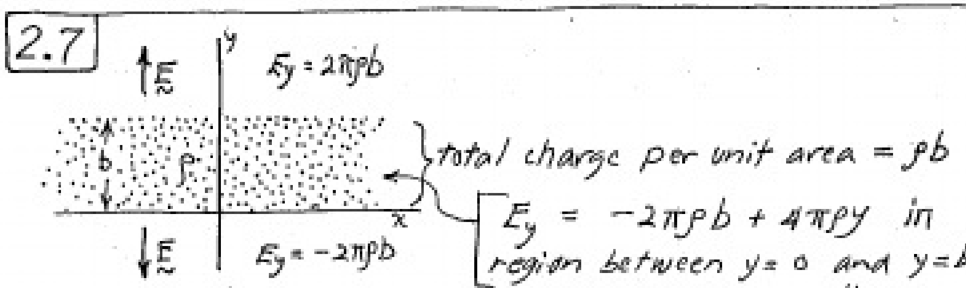
For $r<a$, $\quad \phi = \int_0^r -2\pi\rho r'\,dr' = -\pi\rho r^2$

For $r>a$, $\quad \phi = -\pi\rho a^2 - \int_a^r \frac{2\pi\rho a^2\,dr}{r} = -\pi\rho a^2 - 2\pi\rho a^2 \ln\frac{r}{a}$

(c)



2.7



Set $\phi = 0$ at $y = 0$. Then for $0<y<b$, $\phi(y) = -\int_0^y E_y\, dy'$

$$\phi = -\int_0^y (-2\pi\rho b + 4\pi\rho y')\, dy'$$

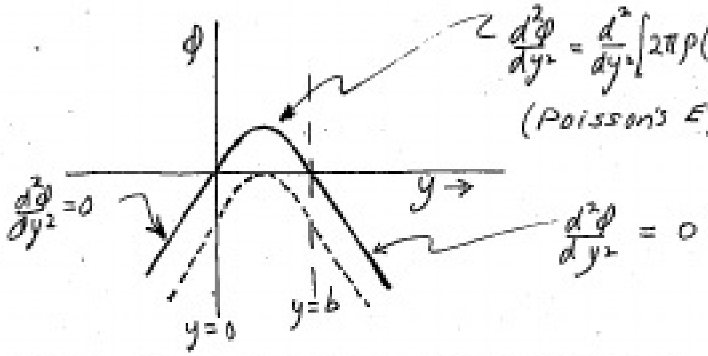
$$= 2\pi\rho \int_0^y b\,dy' - 2y'dy' = 2\pi\rho\,(by - y^2)$$

For $y > b$, $\phi = 0 - \int_b^y 2\pi\rho\, b\, dy' = -2\pi\rho\,(y-b)$.

For $y < 0$ ϕ must have the opposite slope: $\phi = 2\pi\rho y$

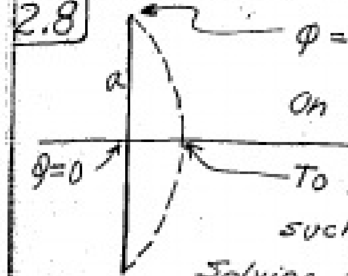
$$\frac{d^2\phi}{dy^2} = \frac{d^2}{dy^2}\left[2\pi\rho(by - y^2)\right] = -4\pi\rho$$

(Poisson's Eq. satisfied)



[Note: We could just as well have taken $\phi = 0$ at $y = \frac{b}{2}$. Then ϕ would be the function shown by -----]

2.8



$\phi = 4\sigma a$ (see Eq. 26, p. 46)

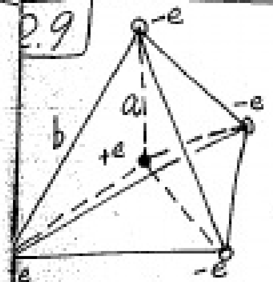
On axis, $\phi = 2\pi\sigma\left[\sqrt{y^2+a^2} - y\right]$ (Eq. 22)

To locate this point, we must find y such that $4\sigma a = 2\pi\sigma\left[\sqrt{y^2+a^2} - y\right]$

Solving for y:

$$\left(\frac{4a}{2\pi} + y\right)^2 = y^2 + a^2 \quad \text{gives} \quad y = a\,\frac{1-\left(\frac{4}{2\pi}\right)^2}{\left(\frac{8}{2\pi}\right)} = 0.466\,a$$

2.9



$b = 10^{-8}$ cm. From the geometry of the regular tetrahedron, $a = \sqrt{\frac{3}{8}}\, b$.

$$U = \frac{1}{2}\sum_j q_j \phi_j$$

[P]otential at one corner electron, due to all other [c]harges $= -\frac{3e}{b} + \frac{e}{a}$

[P]otential at central proton, due to all other [c]harges $= -\frac{4e}{a}$

$$\frac{1}{2}\sum q_j\phi_j = \frac{1}{2}\left[4(-e)\left(-\frac{3e}{b} + \frac{e}{a}\right) + 1(+e)\left(-\frac{4e}{a}\right)\right] = \frac{6e^2}{b} - \frac{4e^2}{a}$$

[N]otice that this is exactly what we would get by summing $q_j q_k / r_{jk}$ over all distinct pairs, as in Eq. 1.8, Chap. 1)

$$= \frac{e^2}{b}\left(6 - \frac{4b}{a}\right) = \frac{e^2}{b}\left(6 - 4\sqrt{\frac{8}{3}}\right) = \frac{e^2}{b}(6 - 6.532) = -\frac{0.532e^2}{b}$$

$$-\frac{0.532\,e^2}{b} = -\frac{.532\,(4.8\times10^{-10})^2}{10^{-8}}\ \text{ergs} = -1.2\times10^{-11}\ \text{ergs} = -7.5\ \text{ev.}$$

[Be]cause the energy is negative, we infer that the [ele]ctrical force on the electrons must be inward; the [str]ucture would contract, its potential energy becoming [ev]en more negative, if the particles were free to

2.10 (a)



Since $d \gg r$ we may assume the charge on each sphere is uniformly distributed over that sphere. Then the potential at the surface of one of the spheres is

$$\varphi = \frac{(Q/2)}{r} + \frac{(Q/2)}{d} \quad \text{(in this approximation).}$$

$$U = \frac{1}{2}\sum q_i \varphi_i = \frac{1}{2}\left[\frac{Q}{2}\left(\frac{Q}{2r} + \frac{Q}{2d}\right) + \frac{Q}{2}\left(\frac{Q}{2r} + \frac{Q}{2d}\right)\right] = \frac{Q^2}{4}\left(\frac{1}{r} + \frac{1}{d}\right)$$

(b) With Q on one sphere, $U = \frac{1}{2}(Q)\left(\frac{Q}{r}\right) = \frac{Q^2}{2r}$

(c) In (a) $\varphi_1 = \varphi_2 = \frac{Q}{2}\left(\frac{1}{r} + \frac{1}{d}\right)$

In (b) $\varphi_1 = \frac{Q}{r}, \quad \varphi_2 = \frac{Q}{d}$

(d) Final charge configuration would be that in (a), the potential energy being thereby lowered by an amount $\frac{Q^2}{4}(1/r - 1/d)$. That amount of energy will have been dissipated in radiation or in the flow of current in the connecting wire.

2.11 $E_x = 6xy \quad E_y = 3x^2 - 3y^2 \quad E_z = 0$

$$(\text{curl } E)_x = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = 0 \qquad (\text{curl } E)_y = \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = 0$$

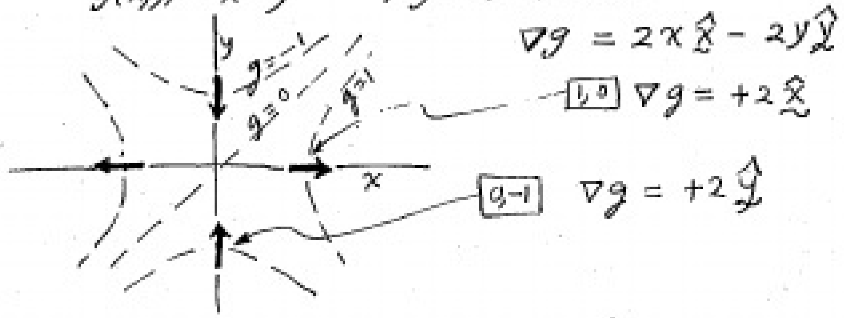
$$(\text{curl } E)_z = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 6x - 6x = 0$$

$$\text{div } E = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 6y - 6y = 0$$

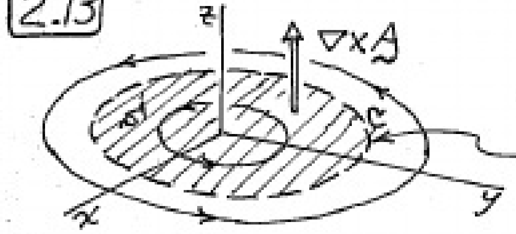
2.12 $f(x,y) = x^2 + y^2 \qquad \nabla^2 f = 2 + 2 \neq 0$

$g(x,y) = x^2 - y^2 \qquad \nabla^2 g = 2 - 2 = 0$

$$\nabla g = 2x\hat{x} - 2y\hat{y}$$



2.13



$\mathbf{A} = -y\hat{x} + x\hat{y}$

$A = \sqrt{y^2 + x^2} = r$

$A = 1$ on unit circle C

$$\int_C \mathbf{A}\cdot d\mathbf{s} = 1\cdot 2\pi = 2\pi$$

$$\nabla\times\mathbf{A} = \hat{x}(0) + \hat{y}(0) + \hat{z}(1+1) = 2\hat{z}$$

$$\int_S (\nabla\times\mathbf{A})\cdot d\mathbf{a} = 2\cdot(\text{area of unit circle}) = 2\pi$$

2.14 (a) $\mathbf{E} = \hat{x}(x+y) + \hat{y}(-x+y) + \hat{z}(-2z)$

2.15 $\nabla\times\mathbf{E} = \hat{x}(0+0) + \hat{y}(0+0) + \hat{z}(-1-1) = -2\hat{z}$

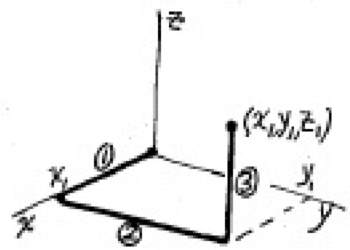
$\nabla\cdot\mathbf{E} = 1 + 1 - 2 = 0$

(b) $\mathbf{G} = \hat{x}(2y) + \hat{y}(2x+3z) + \hat{z}(3y)$

$\nabla\times\mathbf{G} = \hat{x}(3-3) + \hat{y}(0+0) + \hat{z}(2-2) = 0$

$\nabla\cdot\mathbf{G} = 0 + 0 + 0 = 0$

Since $\nabla\times\mathbf{G} = 0$ there exists an f such that $\mathbf{G} = \nabla f$. To determine f, compute the line integral of $\mathbf{G}$ from a fixed point, say $(0,0,0)$, to a general point x_1, y_1, z_1, over any path. Using the path composed of lines ①, ② and ③:



$$\int_{(0,0,0)}^{(x_1,y_1,z_1)} \mathbf{G}\cdot d\mathbf{s} = \int_0^{x_1} G_x(x,0,0)\,dx + \int_0^{y_1} G_y(x_1,y,0)\,dy + \int_0^{z_1} G_z(x_1,y_1,z)\,dz$$

$$= \int_0^{x_1} 0\,dx + \int_0^{y_1} 2x_1\,dy + \int_0^{z_1} 3y_1\,dz = 2x_1y_1 + 3y_1z_1 = f$$

x_1, y_1, z_1 was a general point, so we can drop the subscripts and write: $f = 2xy + 3yz$.

To check that $\nabla f = \mathbf{G}$:

$$\nabla f = 2y\hat{x} + (2x+3z)\hat{y} + 3y\hat{z} = \mathbf{G}$$

(C) $\underset{\sim}{H} = \hat{x}(x^2 - z^2) + \hat{y}(2) + \hat{z}(2xz)$

$\nabla \times \underset{\sim}{H} = \hat{x}(0+0) + \hat{y}(-2z-2z) + \hat{z}(0+0) = -4z\hat{y}$

$\nabla \cdot \underset{\sim}{H} = 2x + 0 + 2x = 4x$

2.16 **2.15** (a) $\nabla \cdot (\nabla \times A) = \frac{\partial}{\partial x}(\nabla \times A)_x + \frac{\partial}{\partial y}(\nabla \times A)_y + \frac{\partial}{\partial z}(\nabla \times A)_z$

$$= \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$= \frac{\partial^2 A_z}{\partial x \partial y} - \frac{\partial^2 A_y}{\partial x \partial z} + \frac{\partial^2 A_x}{\partial y \partial z} - \frac{\partial^2 A_z}{\partial y \partial x} + \frac{\partial^2 A_y}{\partial z \partial x} - \frac{\partial^2 A_x}{\partial z \partial y} = 0,$$

because $\frac{\partial^2 A_z}{\partial x \partial y} = \frac{\partial^2 A_z}{\partial y \partial x}$ for any function A_z with continuous derivatives; likewise for A_x and A_y.

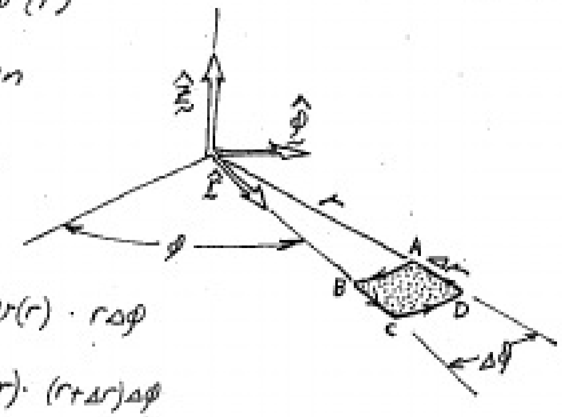
(b) If A is any vector field, $\int_C A \cdot ds \to 0$ over a curve such as C which consists of adjacent paths running in opposite directions. This path bounds the shaded surface S. Hence, by Stokes' theorem,

$\int_S (\nabla \times A) \cdot da = 0$



But the slit is a negligible part of the whole closed surface S', so the same conclusion must apply to S': $\int_{S'} (\nabla \times A) \cdot da = 0$. S' could have been any surface, so we have concluded that the vector function $(\nabla \times A)$ has zero surface integral over any arbitrary closed surface. It follows from Gauss's theorem that $\mathrm{div}(\nabla \times A) = 0$ everywhere, for if the divergence were different from zero in any small region, a surface surrounding just that region would necessarily have a non-zero surface integral. (Of course, we can apply such an argument only to "well-behaved" functions.)

2.16 Given: $\underset{\sim}{v} = \hat{\phi}\, v(r)$



To find the circulation Γ around the path ABCD, to first order in Δr and $\Delta\phi$:

Along AB: $\int v \cdot ds = -v(r) \cdot r\Delta\phi$

Along CD: $\int v \cdot ds = +v(r+\Delta r) \cdot (r+\Delta r)\Delta\phi$

$$= \left(v + \frac{dv}{dr}\Delta r\right)(r+\Delta r)\Delta\phi = \Delta\phi\left[rv + r\frac{dv}{dr}\Delta r + v\Delta r\right]$$

Along BC and DA we get nothing because $\hat{r}\cdot\hat{\phi} = 0$

Thus $\Gamma = \Delta\phi\left[rv + r\frac{dv}{dr}\Delta r + v\Delta r - rv\right]$

Area of patch enclosed $= r\Delta\phi\,\Delta r$

$\lim \frac{\Gamma}{\text{area}} = \frac{v}{r} + \frac{dv}{dr}$ which can be written $\frac{1}{r}\frac{d}{dr}(rv)$

Thus $\text{curl}\, \underset{\sim}{v} = \hat{z}\frac{1}{r}\frac{d}{dr}(rv)$, if $\underset{\sim}{v} = \hat{\phi}\, v(r)$

(a) For a rigid body $v = \omega r$, with ω constant. Then $rv = \omega r^2$ and $\text{curl}\, \underset{\sim}{v} = \hat{z}\left(\frac{1}{r}\right)(2\omega r) = 2\omega\hat{z}$

(b) We want $\frac{1}{r}\frac{d}{dr}(rv) = 0$, which simply requires $rv = \text{constant}$, or $v = \frac{\text{constant}}{r}$

The field looks like this:



Obviously there is a singularity at $r=0$.

(c) Kepler's third law (see Vol. 1, p. 289), in the case of circular motion, is: $r^3\omega^2 = \text{constant}$. Since $v = \omega r$, this can be written $rv^2 = \text{const.}$ or $rv = (\text{const.})\cdot r^{\frac{1}{2}}$, from which

$\text{curl}\, \underset{\sim}{v} = \text{constant} \cdot \hat{z} \cdot r^{-\frac{3}{2}}$

2.17

To derive Eq. 39 from Eq. 43:

$$U = \frac{1}{8\pi}\int E^2\,dv \qquad (36)$$

$$U = \frac{1}{8\pi}\int |\nabla\phi|^2\,dv \qquad (39)$$

$$U = \frac{1}{2}\int \rho\phi\,dv \qquad (43)$$

From Poisson's Eq., $\rho = -\frac{\nabla^2\phi}{4\pi}$

Substituting in (43):

$$U = -\frac{1}{8\pi}\int \phi\nabla^2\phi\,dv$$

Using the identity $\phi\nabla^2\phi = \nabla\cdot(\phi\nabla\phi) - (\nabla\phi)^2$

$$U = \frac{1}{8\pi}\int(\nabla\phi)^2\,dv - \frac{1}{8\pi}\int\nabla\cdot(\phi\nabla\phi)\,dv$$

Consider the second integral extended over a large, but finite, volume V enclosed by a surface S. According to the divergence theorem:

$$\int_V \nabla\cdot(\phi\nabla\phi)\,dv = \int_S (\phi\nabla\phi)\cdot d\mathbf{a}$$

Assume that all sources are contained within some finite region and let S be a gigantic sphere of radius $R \gg$ source dimensions. If Q is the net charge, $\nabla\phi$, which is $-\mathbf{E}$ will vary asymptotically as Q/R^2 as R expands, and ϕ will vary as Q/R. Hence $|\phi\nabla\phi| \sim 1/R^3$. The area of the sphere increases only as R^2. Hence if we let $R\to\infty$, the integral $\int_S(\phi\nabla\phi)\cdot d\mathbf{a} \to 0$.

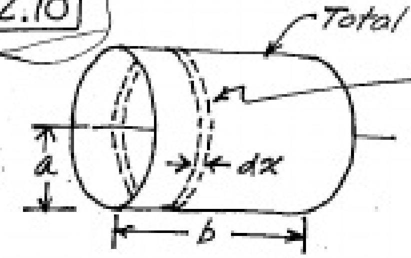


Therefore if we extend the integration over all space, the second integral necessarily gives zero, and we are left with

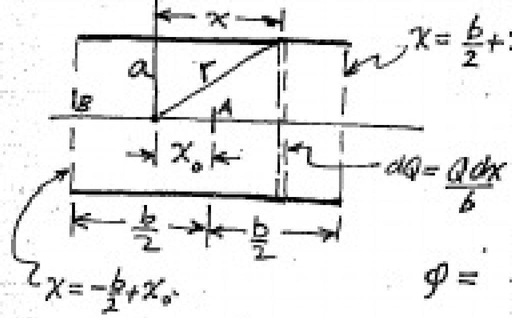
$$U = \frac{1}{8\pi}\int_{\text{all space}}(\nabla\phi)^2\,dv$$

(If the sources were not confined to a finite region we could not be sure that any of these integrals would converge when extended over all space.)

2.18



Total charge Q uniformly distributed. Charge in ring of width dx $= dQ = Q\frac{dx}{b}$. All charge in such a ring is equidistant from a point on the axis, so the ring is a convenient charge element to use in computing the potential at axial points. Let's find potential at a general axial point, distance x_0 from mid-point. Locate origin at the point in question:



$$\phi = \int\frac{dQ}{r} = \int_{-\frac{b}{2}+x_0}^{\frac{b}{2}+x_0}\left(\frac{Q\,dx}{b}\right)\frac{1}{\sqrt{a^2+x^2}}$$

$$\phi = \frac{Q}{b}\Big[\ln\left(\sqrt{a^2+x^2}+x\right)\Big]_{-\frac{b}{2}+x_0}^{\frac{b}{2}+x_0}$$

For potential at mid-point, set $x_0 = 0$:

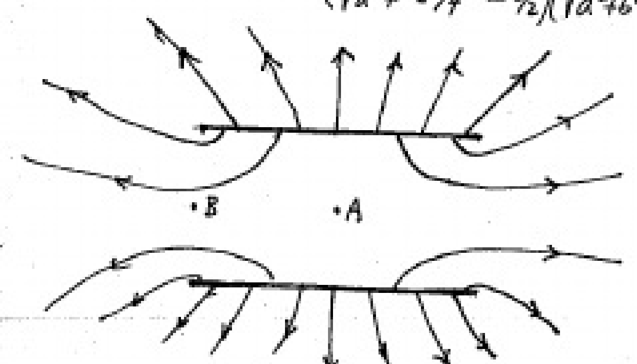
$$\phi_A = \frac{Q}{b}\ln\frac{\sqrt{a^2+b^2/4}+b/2}{\sqrt{a^2+b^2/4}-b/2}$$

For potential at end, set $x_0 = \frac{b}{2}$:

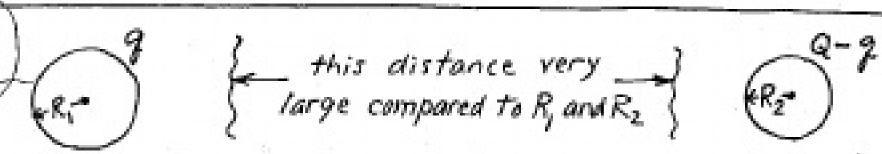
$$\phi_B = \frac{Q}{b}\ln\frac{\sqrt{a^2+b^2}+b}{a}$$

Then

$$\phi_A - \phi_B = \frac{Q}{b}\ln\frac{a\left(\sqrt{a^2+b^2/4}+b/2\right)}{\left(\sqrt{a^2+b^2/4}-b/2\right)\left(\sqrt{a^2+b^2}+b\right)}$$



2.19



$U = \frac{1}{2}\sum q\phi = \frac{q^2}{2R_1} + \frac{(Q-q)^2}{2R_2}$. Condition for minimum is $\frac{dU}{dq} = 0$, or $\frac{q}{R_1} - \frac{Q-q}{R_2} = 0$, which gives $R_2 q = R_1(Q-q)$ or $q = Q\frac{R_1}{R_1+R_2}$

In this case, potential of sphere 1 is

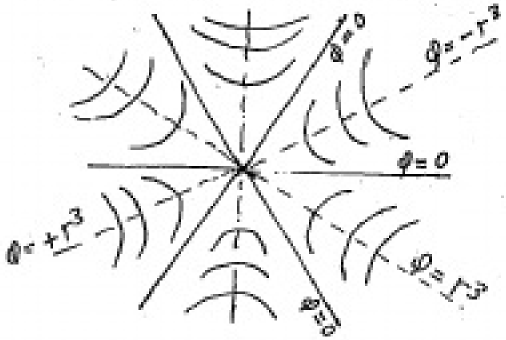
$$\phi_1 = \frac{q}{R_1} = \frac{Q}{R_1+R_2} \; ; \quad \phi_2 = \frac{Q-q}{R_2} = \frac{Q}{R_1+R_2}$$

(same potential)

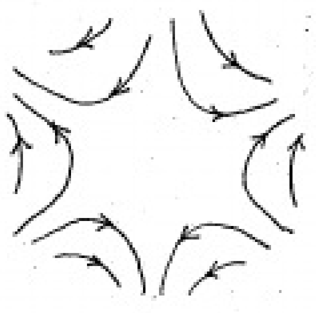
2.20 In Problem 2.1 we found $\phi = y^3 - 3x^2y$

In polar coordinates, $\phi = r^3\sin^3\theta - 3r^3\sin\theta\cos^2\theta$

But $\sin^3\theta - 3\sin\theta\cos^2\theta = -\sin 3\theta$, so that ϕ can be written $\phi = -r^3\sin 3\theta$. This shows that ϕ has 3-fold symmetry around the axis.

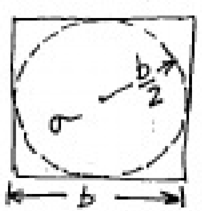


equipotentials



field lines

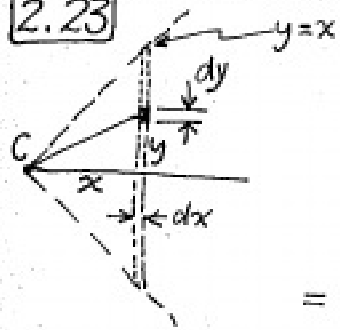
2.22 Divide the square into a disk of radius $b/2$ and four corners. Contribution of disk to potential at center is $2\pi\sigma\left(\frac{b}{2}\right)$ or $\pi\sigma b$. Total charge on corners is:



$$Q_c = \sigma b^2 - \sigma\pi\left(\frac{b}{2}\right)^2 = \sigma b^2\left(1-\frac{\pi}{4}\right) = .2146\sigma b^2$$

This charge is distributed between $r = \frac{b}{2}$ and $r = b/\sqrt{2}$. Hence its contribution to the potential at the center must be less than $\frac{.2146\,\sigma b^2}{.500\,b}$, or $.429\,\sigma b$, but greater than $\frac{.2146\,\sigma b^2}{.707\,b}$ or $.304\,\sigma b$. The total potential must lie between the limits $(3.14 + .30)\sigma b$ and $(3.14 + .43)\sigma b$; $3.44\,\sigma b < \phi < 3.57\,\sigma b$.

2.23



The contribution of the strip to the potential at C is:

$$d\phi = \int_{-x}^{x} \frac{\sigma\,dx}{\sqrt{x^2+y^2}}\,dy = 2\sigma\,dx\int_0^x \frac{dy}{\sqrt{x^2+y^2}}$$

$$= 2\sigma\,dx\left[\ln\left(y+\sqrt{x^2+y^2}\right)\right]_0^x$$

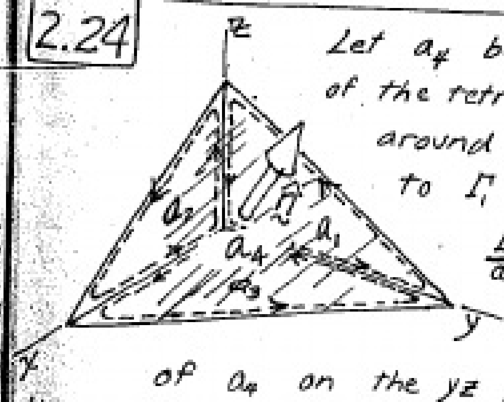
$$= 2\sigma\,dx\,\ln\left(\frac{x+\sqrt{2}x}{x}\right) = 2\ln(1+\sqrt{2})\,\sigma\,dx.$$

The contribution of the whole triangle is then:

$$\int_0^{b/2} 2\ln(1+\sqrt{2})\,\sigma\,dx = \sigma b\ln(1+\sqrt{2}).$$

There are four such triangles: $\phi = 4\ln(1+\sqrt{2})\,\sigma b = 3.525\,\sigma b$

2.24



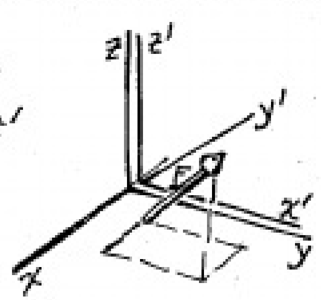
Let a_4 be the area of the sloping face of the tetrahedron. The circulation Γ_4 around this face is obviously equal to $\Gamma_1 + \Gamma_2 + \Gamma_3$. But $\frac{\Gamma_1}{a_1} = \text{"}A_x\text{"}$, $\frac{\Gamma_2}{a_2} = \text{"}A_y\text{"}$, etc.

Now $a_1 = a_4\,\hat{n}\cdot\hat{x}$, for a_1 is just the projection of a_4 on the yz plane. $a_2 = a_4\,\hat{n}\cdot\hat{y}$, etc

Hence $\Gamma_4 = a_1\text{"}A_x\text{"} + a_2\text{"}A_y\text{"} + a_3\text{"}A_z\text{"} = a_4\,\hat{n}\cdot(\hat{x}\text{"}A_x\text{"} + \hat{y}\text{"}A_y\text{"} + \hat{z}\text{"}A_z\text{"})$

Thus Γ_4/a_4 is indeed $\hat{n}\cdot\text{curl}\,\mathbf{F}$

2.25 $\underset{\sim}{F}$ is a vector described, in the xyz frame by the components $F_x \; F_y \; F_z$. In the rotated $x'y'z'$ frame the <u>same</u> <u>vector</u> is described by the components $F_{x'} \; F_{y'} \; F_{z'}$. We see that



$$F_{x'} = F_y \qquad F_{y'} = -F_x \qquad F_{z'} = F_z$$

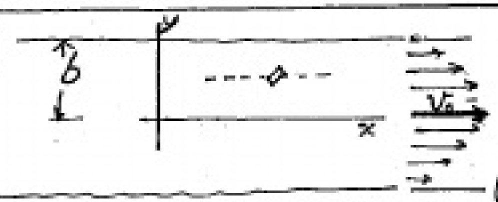
Similarly, $\hat{\underset{\sim}{x}}' = \hat{\underset{\sim}{y}} \qquad \hat{\underset{\sim}{y}}' = -\hat{\underset{\sim}{x}} \qquad \hat{\underset{\sim}{z}}' = \hat{\underset{\sim}{z}}$

Also, if f is any scalar field,

$$\frac{\partial f}{\partial x'} = \frac{\partial f}{\partial y} \qquad \frac{\partial f}{\partial y'} = -\frac{\partial f}{\partial x} \qquad \frac{\partial f}{\partial z'} = \frac{\partial f}{\partial z}$$

Now if $\hat{x}\frac{\partial F_x}{\partial x} + \hat{y}\frac{\partial F_y}{\partial y} + \hat{z}\frac{\partial F_z}{\partial z}$ represents a vector, then $\hat{x}'\frac{\partial F_{x'}}{\partial x'} + \hat{y}'\frac{\partial F_{y'}}{\partial y'} + \hat{z}'\frac{\partial F_{z'}}{\partial z'}$ ought to represent the same vector. But, it <u>doesn't</u>, for the part $\hat{y}'\frac{\partial F_{y'}}{\partial y'}$ expressed in xyz coordinates, turns out to be $-\hat{x}\frac{\partial F_x}{\partial x}$, since $\hat{y}' = -\hat{x}$, $F_{y'} = -F_x$ and $\frac{\partial}{\partial y'} = -\frac{\partial}{\partial x}$.

2.26



$$v_x = v_0\left(1 - \frac{y^2}{b^2}\right)$$
$$v_y = 0$$
$$(\nabla \times \underset{\sim}{v})_z = -\frac{\partial v_x}{\partial y} = \frac{2v_0 y}{b^2}$$

From the result of Problem 2.16 (a), for a rigid body rotating with angular speed ω, the particle velocities form a vector field with $|\nabla \times \underset{\sim}{v}| = 2\omega$. We may expect the chip to rotate so that the velocity field of the points on its bottom, call it $\underset{\sim}{v}_c$, matches the local velocity field of the water, call it $\underset{\sim}{v}_w$, as closely as it can. Now $\underset{\sim}{v}_w$ is not, even locally, a field of pure rotation + translation. It is rather pure shear + rotation. Nevertheless, if $\nabla \times \underset{\sim}{v}_c = \nabla \times \underset{\sim}{v}_w$, the difference between $\underset{\sim}{v}_c$ and $\underset{\sim}{v}_w$ will be a field with zero curl — and that will not, in general, affect the rotation of the chip. Our condition is therefore simply

$$2\omega = \frac{2v_0 y}{b^2}, \quad \text{or} \quad \omega = \frac{v_0 y}{b^2} = \frac{v_0}{2b} \quad \text{for } y = b/2$$

At the same time the chip is being carried downstream with a velocity $v_0\left(1 - \frac{(b/2)^2}{b^2}\right) = \frac{3}{4}v_0$

The distance the chip travels while rotating through 360° is therefore

$$S = \frac{2\pi}{\left(\frac{v_0}{2b}\right)} \cdot \frac{3}{4}v_0 = 3\pi b$$

2.27 (a) Explained in the figure.

(b) $$\frac{1}{r}\frac{d^2}{dr^2}(rV) = \frac{1}{r}\frac{d}{dr}\left(V + r\frac{dV}{dr}\right)$$
$$= \frac{1}{r}\left(2\frac{dV}{dr} + r\frac{d^2V}{dr^2}\right) = \frac{2}{r}\frac{dV}{dr} + \frac{d^2V}{dr^2}$$

(c) If $\nabla^2 V = 0$, where $V = V(r)$, then

$\frac{1}{r}\frac{d^2}{dr^2}(rV) = 0$, implying: $\frac{d}{dr}(rV) = C_1$ (a constant)

Then $rV = C_1 r + C_2$ or $V = C_1 + \frac{C_2}{r}$

(d) $V = \frac{e^{-\lambda r}}{r} \qquad \frac{1}{r}\frac{d^2}{dr^2}rV = \frac{\lambda^2 e^{-\lambda r}}{r} = \lambda^2 V$

Thus $\nabla^2 V - \lambda^2 V = 0$ for this $V(r)$.

2.28 $V = \frac{Ce^{-\lambda r}}{r} \qquad F = -\frac{dV}{dr} = C\left(-\frac{\lambda}{r} - \frac{1}{r^2}\right)e^{-\lambda r} = -\frac{Ce^{-\lambda r}}{r^2}(1+\lambda r)$

$(1+\lambda r)e^{-\lambda r} = (1+\lambda r)\left(1 - \lambda r + \frac{\lambda^2 r^2}{2} - \ldots\right) = 1 - \frac{\lambda^2 r^2}{2} + \ldots$

Hence increasing λ from zero reduces magnitude of force.

Poisson's Eq. is replaced by

$$\nabla^2 V - \lambda^2 V = -4\pi\rho$$

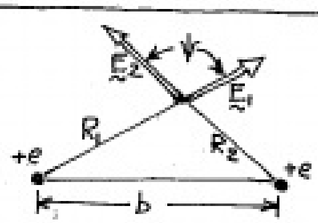
for Yukawa sources. We have already shown (Prob. 27(d)) that in a source free region V satisfies $\nabla^2 V - \lambda^2 V = 0$. Now consider part of a source that is small in dimensions compared to $1/\lambda$. In this region the Yukawa field of the local sources is indistinguishable from a Coulomb field, so $\nabla^2 V_\ell = -4\pi\rho$ where V_ℓ is the contribution of just these sources to the potential. The more distant sources contribute $V_d = V - V_\ell$, for which $\nabla^2 V_d - \lambda^2 V_d = 0$. The total field $V = V_d + V_\ell$ therefore satisfies the above equation.

2.29 For force $\sim r^{-1.5}$, arrange a roughly spherical "cage" of negative charges. This will hold a positive particle stably at the center. To see that, recall that the field would be practically zero for a force $\sim r^{-2}$, <u>everywhere</u> in the cage. With the $r^{-1.5}$ force, as we move the positive charge from C to A, the attraction it feels from the negative charges on the left <u>inc</u>reases less rapidly, while the attraction from the negative charges on the right <u>dec</u>reases less rapidly than in the Coulomb case. If the total force remains zero in the Coulomb case, as we know it does, then in this case a force must arise directed backward toward the center. For $F \sim r^{-2.5}$, use a cage of positive charges to hold the positive charge, and reverse the argument just given.



2.30 We are concerned with he integral

$$\frac{1}{4\pi}\int_{\text{all space}} \mathbf{E}_1\cdot\mathbf{E}_2\, dv$$

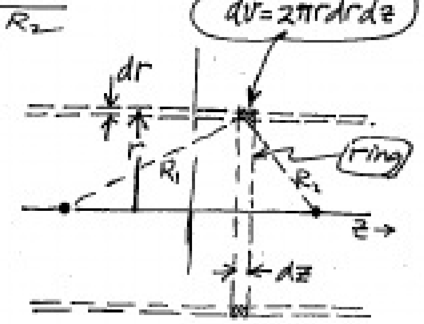


here $\mathbf{E}_1$ and $\mathbf{E}_2$ are the fields f the point charges. $\mathbf{E}_1\cdot\mathbf{E}_2 = \frac{e^2}{R_1^2R_2^2}\cos\psi$

sing "law of cosines":

$$\cos\psi = \frac{R_1^2+R_2^2-b^2}{2R_1R_2}$$

le could set up the integral in ylindrical coordinates as follows: ith $dv = 2\pi r\,dr\,dz$,



$$\int_{\text{all space}} \mathbf{E}_1\cdot\mathbf{E}_2\, dv = \frac{e^2}{4\pi}\int_0^\infty 2\pi r\,dr\int_{-\infty}^{\infty}\frac{R_1^2+R_2^2-b^2}{2R_1^3R_2^3}\,dz$$

$R_1^2 = r^2 + (\frac{b}{2}+z)^2 \qquad R_2^2 = r^2 + (\frac{b}{2}-z)^2$ Introduce imensionless variables $u = z/b$, $w = r/b$. Then

$$R_1^2+R_2^2-b^2 = 2r^2 - \frac{b^2}{2} + 2z^2 = b^2(2w^2+2u^2-\tfrac{1}{2})$$

$$R_1^3R_2^3 = b^6\left[w^2+(\tfrac{1}{2}+u)^2\right]^{3/2}\left[w^2+(1/2-u)^2\right]^{3/2}$$

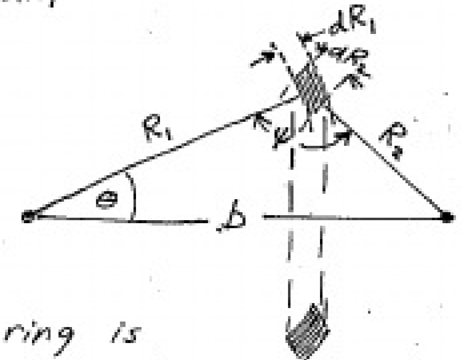
$$\int \mathbf{E}_1\cdot\mathbf{E}_2\, dv = \frac{e^2}{4b}\int_0^\infty w\,dw\int_0^\infty\frac{(4w^2+4u^2-1)\,du}{\left[w^2+(\frac{1}{2}+u)^2\right]^{3/2}\left[w^2+(\frac{1}{2}-u)^2\right]^{3/2}}$$

he integral has been written out in this form only to ow explicitly that it reduces to $\frac{e^2}{b}$ times a definite tegral whose value (if it converges!) must be a number ependent of b. Of course, a dimensional argument would e led us to the same conclusion. Now the work required to ing the two charges from infinite separation in to the ition shown is, we know, just e^2/b. Therefore, if $\frac{1}{4\pi}\int \mathbf{E}_1\cdot\mathbf{E}_2\,dv$ is the correct formula for the field energy exclusive of "self-energy", the integral $\int_0^\infty w\,dw\int_0^\infty \frac{\sim\sim\sim}{\sim\sim\sim}\,du$ must have the value 1. Conversely,

an evaluation of the integral with that result would serve to establish the field energy formula for two charges, and thence, by superposition, for any system of charges.

If one has to evaluate an integral of this sort, there is a better way to set it up, using R_1 and R_2 as the variables. Consider the ring-shaped volume element in the figure below. Its cross section is a little parallelogram of altitude dR_1





and base $\frac{dR_2}{\sin\psi}$.

The radius of the ring is $R_1 \sin\theta$. Thus the volume dv is $dR_1 \frac{dR_2}{\sin\psi}(R_1 \sin\theta \cdot 2\pi)$ But $\frac{\sin\theta}{\sin\psi} = \frac{R_2}{b}$ (law of sines)

so we get $dv = 2\pi \frac{R_1 R_2 dR_1 dR_2}{b}$ and our integral becomes:

$$\frac{1}{4\pi}\int \underset{\sim}{E_1} \cdot \underset{\sim}{E_2}\, dv = \frac{e^2}{4\pi}\int \frac{2\pi R_1 R_2\, dR_1 dR_2}{b}\left(\frac{R_1^2 + R_2^2 - b^2}{2R_1^3 R_2^3}\right)$$

$$= \frac{e^2}{4b}\int \frac{(R_1^2 + R_2^2 - b^2)\, dR_1 dR_2}{R_1^2 R_2^2}$$

But how do we sweep out all space? Hold R_1 constant, $R_1 < b$ and let R_2 sweep over the spherical shell between R_1 and $R_1 + dR_1$.

Then R_2 goes from $b - R_1$ to $b + R_1$.

For $R_1 > b$, R_2 goes from $R_1 - b$ to $R_1 + b$:



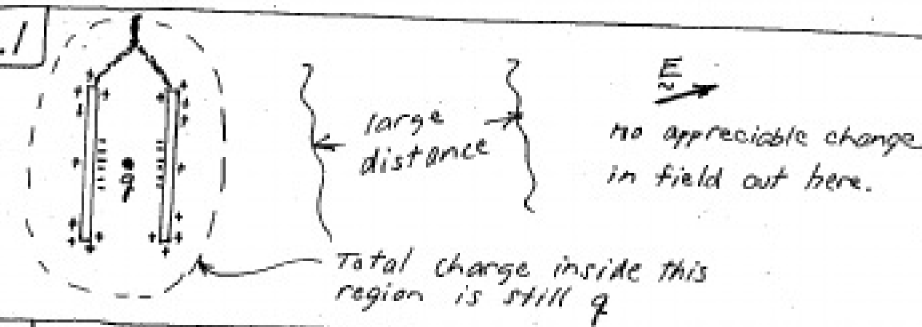
$$\frac{1}{4\pi}\int \underset{\sim}{E_1} \cdot \underset{\sim}{E_2}\, dv =$$

$$\frac{e^2}{4b}\left\{ \int_0^b dR_1 \int_{b-R_1}^{b+R_1} \frac{(R_1^2 + R_2^2 - b^2)\, dR_2}{R_1^2 R_2^2} + \int_b^\infty dR_1 \int_{R_1-b}^{R_1+b} \frac{(R_1^2 + R_2^2 - b^2)\, dR_2}{R_1^2 R_2^2} \right\}$$

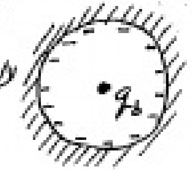
The integral is now elementary!

CHAPTER 3

3.1

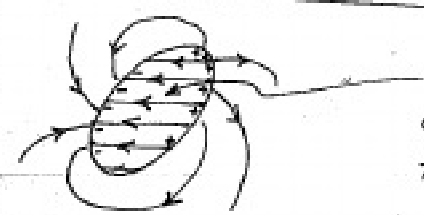


3.2 The force on q_b is zero. The field inside the spherical cavity is quite independent of anything outside. A charge $-q_b$ is uniformly distributed over the conducting surface. The same goes for q_c.



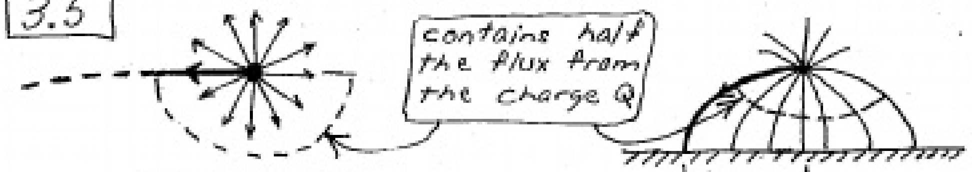
Since the total charge on conductor A is zero, a charge of amount $q_b + q_c$ must be distributed over its external surface. If q_d were absent, the field outside A would be the symmetrical, radial field, $E = \frac{q_b + q_c}{r^2}$, the same as the field of a point charge located where the center of the sphere is located. The influence of q_d will slightly alter the distribution of charge on A, but without affecting the total amount. Hence for large r, the force on q_d will be approximately $q_d \frac{(q_b + q_c)}{r^2}$. The force on A must be precisely equal and opposite to the force on q_d. The exact value of the force on q_d is the sum of the force just given, $q_d(q_b + q_c)/r^2$, and the force that would act on q_d if the total charge on and within A were zero.]

3.3



uniform field inside, equal and opposite to the field in Fig. 3.1 (a)

3.5

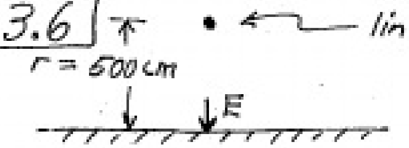


Using Eqs 8 and 9, p. 93, we determine R so that half of $-Q$, the induced charge on the plane, is contained within the circle of radius R:

$$-\frac{Q}{2} = \int_0^R \sigma \cdot 2\pi r\,dr, \text{ or } \frac{1}{2} = \int_0^R \frac{hr\,dr}{(h^2+R^2)^{3/2}} = \left[\frac{-h}{\sqrt{h^2+R^2}}\right]_0^R = 1 - \frac{h}{\sqrt{h^2+R^2}}$$

Then $\frac{h}{\sqrt{h^2+R^2}} = \frac{1}{2}$, or $h^2 + R^2 = 4h^2$, or $R = \sqrt{3}\,h$

3.6



• ← line charge $\lambda = 10^3$ esu/cm

$$E = \underbrace{\frac{2\lambda}{r}}_{\text{line}} + \underbrace{\frac{2\lambda}{r}}_{\text{image}} = \frac{4\times10^3}{500} = 8\ \frac{\text{statvolts}}{\text{cm}}$$

○ ← "image" of line, $\lambda = -10^3$ esu/cm

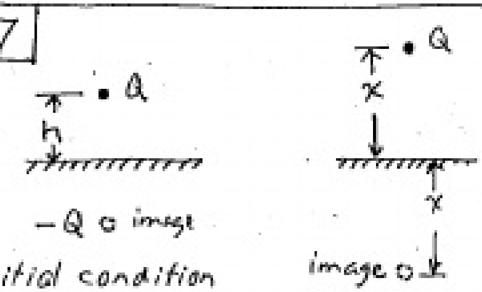
The force on the wire, per unit length, is that which a line charge in the position of the image would cause:

$$\text{force/unit length} = \lambda\left(\frac{-2\lambda}{2r}\right) = \frac{10^6}{500} = 2\times10^3 \text{ dynes/cm (downward)}$$

(charge on wire per unit length → λ; field at wire due to image → $\left(\frac{-2\lambda}{2r}\right)$)

3.7

3.5



Force required to move charge is $F = \frac{Q^2}{(2x)^2}$

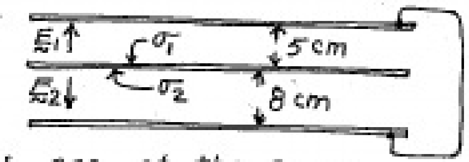
The second student calculates as follows:

Work done $= \int F\,dx = \int_h^\infty \frac{Q^2}{4x^2}dx = \frac{Q^2}{4h}$ This is the correct answer, not $\frac{Q^2}{2h}$. Note that if two real charges, Q and $-Q$ were pulled apart symmetrically, the total work done would be $Q^2/2h$, but the agency moving Q would only supply half of it.

3.8

Given: $\sigma_1 + \sigma_2 = 10$ esu/cm²

$E_1 = 4\pi\sigma_1$ $E_2 = 4\pi\sigma_2$

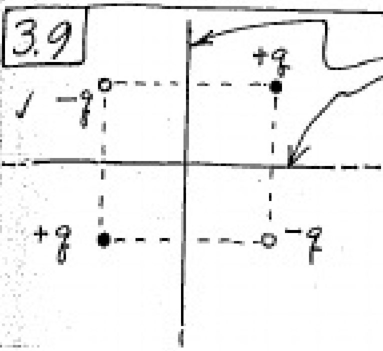


since top and bottom plates are at the same potential, $E_1 \cdot 5\text{ cm} = E_2 \cdot 8\text{ cm}$, or $5\sigma_1 = 8\sigma_2$

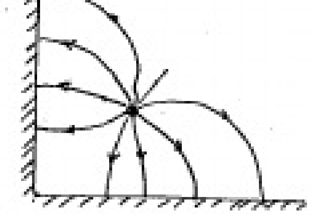
$$\left.\begin{array}{l}\sigma_1 + \sigma_2 = 10\\ 5\sigma_1 - 8\sigma_2 = 0\end{array}\right\} \text{ solving for } \sigma_1 \text{ and } \sigma_2:$$

$\sigma_1 = 6.15$, $\sigma_2 = 3.85$ $\frac{\text{esu}}{\text{cm}^2}$

3.9



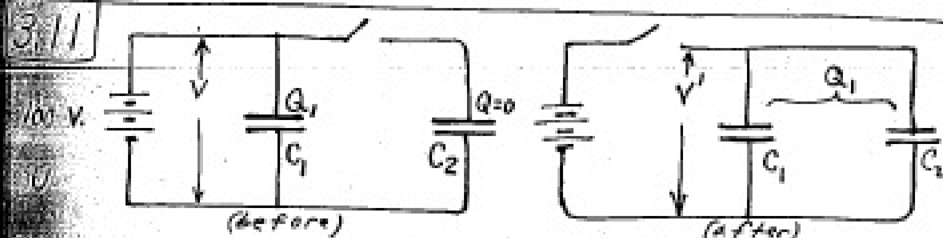
Planes perpendicular to paper. $\phi = 0$ at every point on these two planes, because for any such point, $\sum \frac{q}{r} = 0$, by symmetry.





The method won't work unless $\theta = \pi/N$, N an integer. We have to divide space into an even number of similar wedges, in order to have an alternating ring of images. It won't work, for example, for $\theta = 120°$.

3.11



Given: $V = 100$ volts; $C_1 = 100\,pF = 10^{-10}$ farads; $V' = 30$ volts

$Q_1 = VC_1 = 10^{-8}$ coulomb $= V'(C_1 + C_2) = 30(10^{-10} + C_2)$

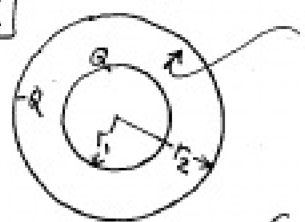
$C_2 = \frac{7}{3}C_1 = 233\,pF$ Energy stored initially: $\frac{1}{2}Q_1V$

Energy remaining: $\frac{1}{2}Q_1V'$ Energy lost: $\frac{1}{2}Q_1(V - V')$

$= \frac{1}{2} \cdot 10^{-8}$ coulomb $\cdot$ 70 volts $= 35\times10^{-7}$ joule. This energy

was dissipated in the resistance in the circuit and to some extent, perhaps, in electromagnetic radiation. If the circuit resistance is fairly low, as it would ordinarily be, the transfer of charge from C_1 to C_2 is oscillatory, and the lower the resistance, the longer the oscillatory current persists.

3.12



$$E = \frac{Q}{r^2}$$

$$\phi_{12} = \int_{r_1}^{r_2} \frac{Q}{r^2}\,dr = Q\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$C = \frac{Q}{\phi_{12}} = \frac{1}{\frac{1}{r_1} - \frac{1}{r_2}} = \frac{r_1 r_2}{r_2 - r_1}$$

If $r_2 - r_1 \ll r_1$ the capacitor should be approximately equivalent to a flat-plate capacitor of area $4\pi r^2$ (where $r \approx r_1 \approx r_2$) and plate separation $r_2 - r_1$. Indeed, if we use for r the geometric mean of r_1 and r_2, the equivalence is exact:



$$\frac{1}{4\pi} \cdot \frac{4\pi\left(\sqrt{r_1 r_2}\right)^2}{r_2 - r_1} = \frac{r_1 r_2}{r_2 - r_1} = C \text{ as calculated above}$$

3.13 $E = \frac{Q}{r^2}$, $r_1 < r < r_2$; $E = 0$ elsewhere

$$\frac{1}{8\pi}\int_{\text{all space}} E^2\,dv = \frac{1}{8\pi}\int_{r_1}^{r_2} \frac{Q^2}{r^4}\,4\pi r^2\,dr$$

$$= \frac{1}{2}Q^2\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}C\phi_{12}^2$$

3.14



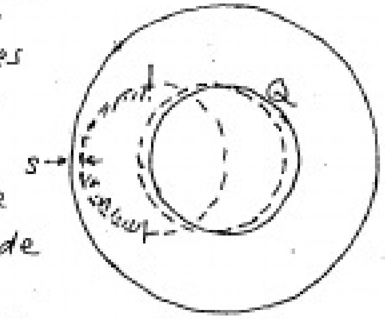
$\phi = Q/R$ $\quad C = \frac{Q}{\phi} = R$

radius of earth $= 6.4 \times 10^8$ cm

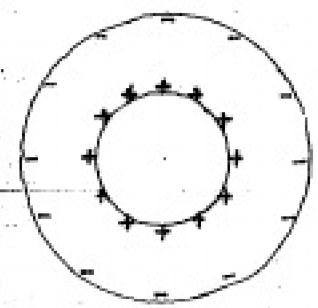
$1\ \mu\text{fd} = (9 \times 10^{11}) \times 10^{-6}$ cm $= 9 \times 10^5$ cm

$$C_{\text{earth}} = \frac{6.4 \times 10^8 \text{ cm}}{9 \times 10^5 \frac{\text{cm}}{\mu\text{fd}}} = 710\ \mu\text{fd}$$

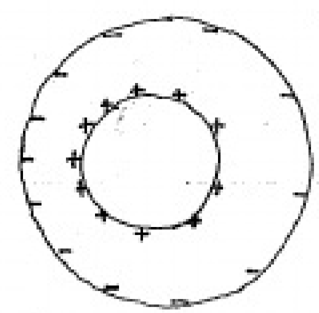
3.15 The stored energy is $\frac{1}{2}\frac{Q^2}{C}$. For constant Q the energy decreases if C increases. It is clear from the symmetry that C is either maximum or minimum when the spheres are concentric. To decide which, imagine the inner sphere displaced until it almost touches the outer sphere.



In the position shown, the left half of the inner sphere obviously contributes more to the capacitance than the right half, and as $s \to 0$ this contribution increases. So we expect C to increase steadily as $s \to 0$.* This is not a rigorous proof that C is a minimum for the central position, but if it isn't, it must have a minimum at some intermediate location, which seems highly unlikely. If the central position is a minimum in C, it is a maximum in potential energy and therefore an unstable equilibrium position. Another argument confirms this: consider how the charge distribution changes as the inner conductor is moved off center:



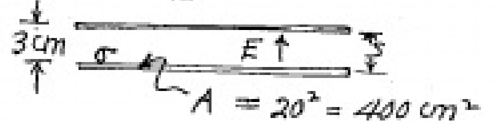
charge distributions symmetrical; net force zero.



charges concentrated in region where conductors are closer together; net force to left on inner sphere.

*One can show, in fact, that C grows infinite like $\ln(\frac{1}{s})$ as $s \to 0$.

3.16 $\Phi_{12} = 10$ statvolts $\qquad E = \dfrac{10\text{ statvolts}}{3\text{ cm}} = \dfrac{10}{3}\ \dfrac{\text{statvolts}}{\text{cm}}$



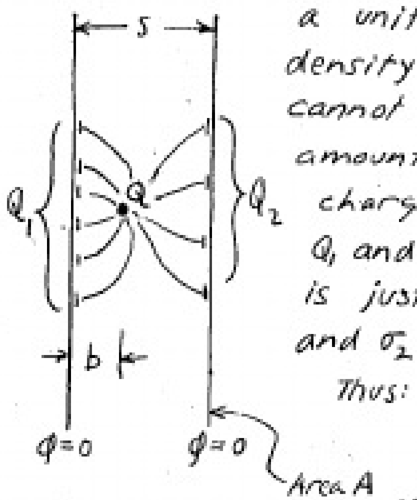
$$\sigma = \frac{E}{4\pi}$$

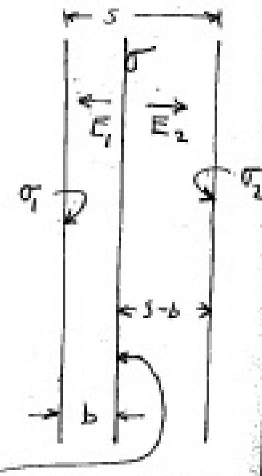
Force on one plate $= \frac{1}{2} E\sigma \times \text{area} = \frac{1}{2}\cdot E \cdot \frac{E}{4\pi}\cdot A$

$$= \frac{1}{8\pi}E^2 A = \frac{1}{8\pi}\cdot\left(\frac{10}{3}\right)^2\cdot 400 = 177\text{ dynes}$$

If plates come together while charge remains constant, then force is constant and work that can be done is $Fs = \frac{1}{8\pi}E^2 As = \frac{1}{8\pi}E^2(\text{Volume})$. $= (177\text{ dynes})(3\text{ cm}) = 531$ ergs, the energy originally stored in the field.

3.17 Imagine charge Q smeared out in a uniform sheet of density $\sigma = \frac{Q}{A}$. This cannot affect the total amount of induced charge on each plate, Q_1 and Q_2. In fact σ_1 is just Q_1 smeared out and σ_2 is Q_2 smeared out. Thus: $\dfrac{Q_1}{Q_2} = \dfrac{\sigma_1}{\sigma_2}$





Q smeared out as a sheet.

But from Gauss's law, $\sigma_1 = E_1/4\pi$ and $\sigma_2 = E_2/4\pi$. Also, because the two plates are at the same potential, $E_1 b = E_2(s-b)$. Hence:

$$\frac{Q_1}{Q_2} = \frac{\sigma_1}{\sigma_2} = \frac{E_1}{E_2} = \frac{s-b}{b}.$$

Therefore, with due regard to sign, we have: $Q_1 = -Q\left(\frac{s-b}{s}\right) \qquad Q_2 = -Q\frac{b}{s}$.

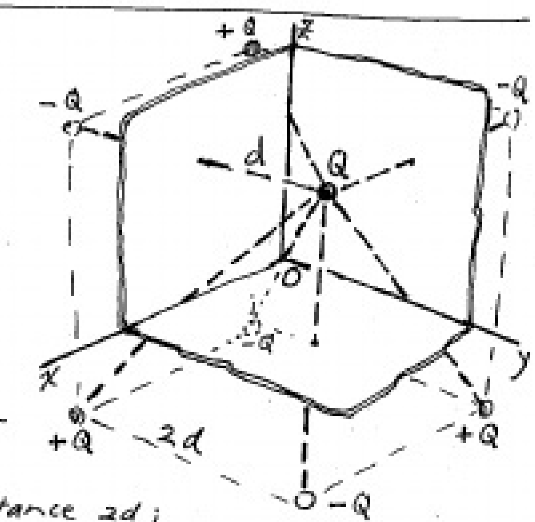
3.18 $(\text{Potential})^2 \sim \left(\dfrac{\text{charge}}{\text{distance}}\right)^2 \sim \dfrac{\text{charge}^2}{\text{distance}^2} \sim \text{force}$

$$(1\text{ statvolt})^2 = \frac{(1\text{ esu})^2}{(1\text{ cm})^2} = 1\text{ dyne}$$

$$(1\text{ megavolt})^2 = \left(\frac{10^6}{300}\text{ statvolt}\right)^2 = \frac{10^8}{9}\text{ dynes}$$

$$= \frac{10^8}{9\times 453\times 980}\text{ pounds} \simeq 25\text{ pounds}$$

3.19 "Image" charges are at the other seven corners of the cube of side $2d$. From the symmetry it is obvious that the direction of the resultant force on Q is parallel to the "body diagonal" OQ. Hence we need compute only force components in that direction:



(a) 3 negative charges at distance $2d$; direction makes angle $\cos^{-1}\frac{1}{\sqrt{3}}$ with OQ:

$$F = 3\cdot\frac{Q^2}{(2d)^2}\cdot\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{4}\frac{Q^2}{d^2}\quad\text{(toward origin)}$$

(b) 3 positive charges at distance $2\sqrt{2}\,d$; direction makes angle $\cos^{-1}\sqrt{\frac{2}{3}}$ with OQ:

$$F = 3\frac{Q^2}{(2\sqrt{2}d)^2}\sqrt{\frac{2}{3}} = \frac{\sqrt{3}}{4\sqrt{2}}\frac{Q^2}{d^2}\quad\text{(away from origin)}$$

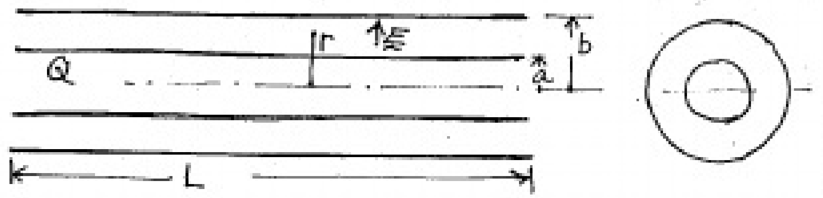
(c) 1 negative charge at opposite corner; distance $\sqrt{3}\cdot 2d$:

$$F = \frac{Q^2}{(2\sqrt{3}d)^2} = \frac{Q^2}{12d^2}.\quad\text{(toward origin)}$$

Total force on $Q = \dfrac{Q^2}{d^2}\left(\dfrac{\sqrt{3}}{4} - \dfrac{\sqrt{3}}{4\sqrt{2}} + \dfrac{1}{12}\right) = \dfrac{0.210\,Q^2}{d^2}$ toward origin.

3.20

(a)



Neglecting end effect, we assume charge Q is uniformly distributed along cylinder. Then field E is that of an axial line charge of density $\lambda = Q/L$. That is, $E = \frac{2\lambda}{r} = \frac{2Q}{rL}$. The potential difference ϕ_{ab} is: $\phi_{ab} = \int_a^b \frac{2Q}{L}\frac{dr}{r} = \frac{2Q}{L}\ln\frac{b}{a}$

Since $Q = C\phi_{ab}$, the capacitance C is given by

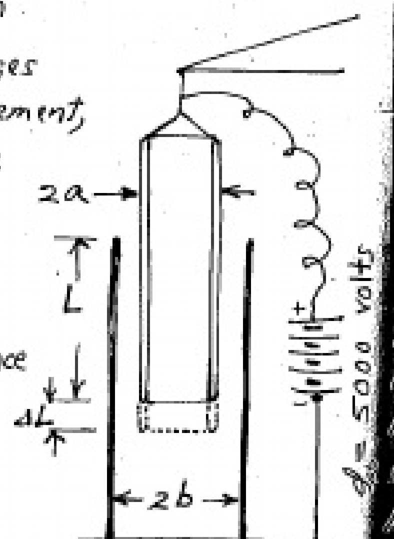
$$C = \frac{L}{2\ln\left(\frac{b}{a}\right)}$$

If $b-a \ll a$, $\ln\left(\frac{b}{a}\right) = \ln\left(1+\frac{b-a}{a}\right) \approx \frac{b-a}{a}$

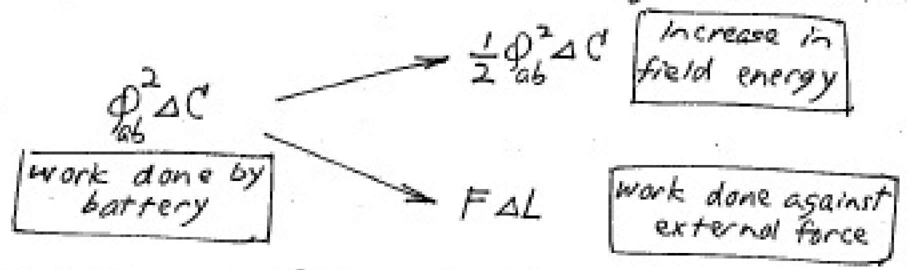
Then $C \approx \frac{aL}{2(b-a)}$ But $2\pi aL$ is the area of the inner cylinder and $b-a$ is the plate separation, so this is just what we would get using the formula for the parallel-plate capacitor: $C = \frac{1}{4\pi}\frac{\text{area}}{\text{separation}}$.

(b) Consider the energy changes involved in a downward displacement, by ΔL, of the inner cylinder. The capacitance increases from $C = \frac{L}{2\ln\frac{b}{a}}$ to $C + \Delta C = \frac{L+\Delta L}{2\ln\frac{b}{a}}$.

With constant potential difference ϕ_{ab}, the stored electrical energy $\frac{1}{2}C\phi_{ab}^2$ increases by $\frac{\Delta C}{2}\phi_{ab}^2$.



At the same time, an amount of charge $\Delta Q = \phi_{ab}\Delta C$ flows into the capacitor. The battery thereby does work, in amount $\phi_{ab}\Delta Q = \phi_{ab}^2\Delta C$. This is twice the increase in stored energy in the field. The difference is the work done against the external force F which balances the electrical attraction of the cylinders. That is:

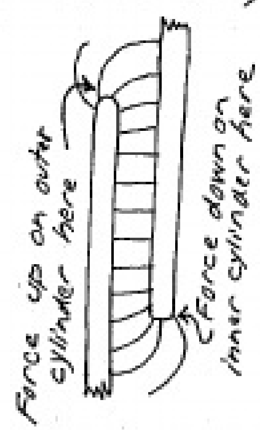


Hence $F\Delta L = \frac{1}{2}\phi_{ab}^2\Delta C$ from which we get

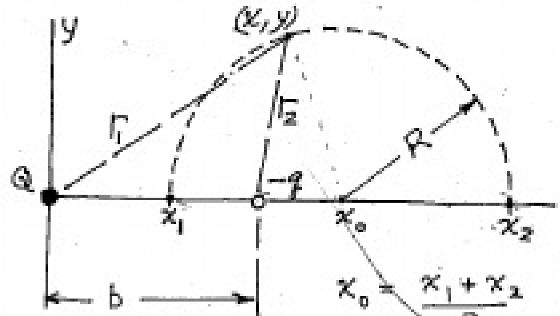
$F = \frac{1}{2}\phi_{ab}^2\frac{\Delta C}{\Delta L}$ This is a quite general formula. In the case at hand $\frac{\Delta C}{\Delta L} = \frac{1}{2\ln\frac{b}{a}}$.

With $b/a = 3/2$ and $\phi_{ab} = 5000$ volts $= 16.7$ statvolts we get $F = \frac{1}{2}(16.7)^2\frac{1}{.812} = 171$ dynes

Note: End effects do not spoil the accuracy of this simple result, for the downward displacement of the inner cylinder leaves the end fields themselves unaltered; it simply lengthens the region where the field is nicely cylindrical. That is, we can safely ignore the end fields in calculating dC/dL, even when they would seriously affect C itself. Nevertheless, the origin of the force just calculated lies in the very end fields that our method permits us to ignore, for it is only at the ends that we find vertical components of the electric field.



3.22 (a) Positive charge Q at origin; negative charge $-q$ at $x=b$, $y=0$. Locate x_1 and x_2 where potential $\phi = 0$:



$x_0 = \frac{x_1+x_2}{2}$

$R = \frac{x_2-x_1}{2}$

$\phi = 0$ at x_1 if

$\frac{Q}{x_1} + \frac{-q}{b-x_1} = 0$. Solving for x_1: $q x_1 = Q(b-x_1)$, or $x_1 = \frac{bQ}{Q+q}$

$\phi = 0$ at x_2 if

$\frac{Q}{x_2} + \frac{-q}{x_2-b} = 0$; Solving for x_2 we get $x_2 = \frac{bQ}{Q-q}$

A sphere passing through x_1 and x_2 will have its center at $x_0 = \frac{x_1+x_2}{2} = \frac{bq}{2}\left(\frac{1}{Q+q} + \frac{1}{Q-q}\right) = \frac{bQ^2}{Q^2-q^2}$

Its radius is: $R = \frac{x_2-x_1}{2} = \frac{bq}{2}\left(\frac{1}{Q+q} + \frac{1}{Q-q}\right) = \frac{bQq}{Q^2-q^2} = x_0\frac{q}{Q}$

We want to show that if xy is a point on this sphere, $\frac{r_1}{r_2} = \frac{Q}{q}$. If so, ϕ will be zero everywhere on the sphere. If (x,y) is on the circle of radius R, we have: $(x-x_0)^2 + y^2 = R^2$, the equation of the circle.

$r_1^2 = x^2 + y^2 = R^2 - x_0^2 + 2xx_0$ But $x_0^2 - R^2$

$= x_0^2\left(1 - \frac{q^2}{Q^2}\right) = \frac{b^2Q^4}{(Q^2-q^2)^2}\frac{(Q^2-q^2)}{Q^2} = \frac{b^2Q^2}{Q^2-q^2}$

Thus $r_1^2 = (2xb - b^2)\frac{Q^2}{(Q^2-q^2)}$

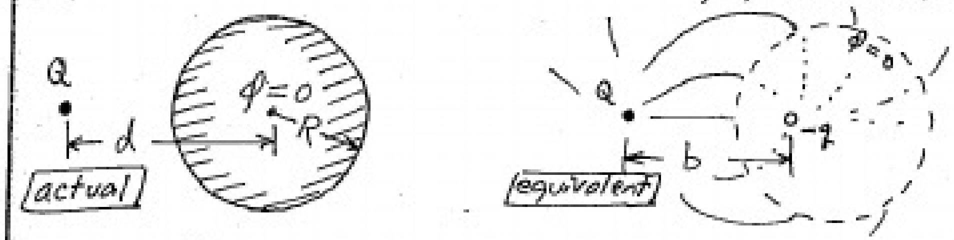
$r_2^2 = (b-x)^2 + y^2 = x^2 + y^2 - (2xb - b^2)$

$= (2xb - b^2)\left[\frac{Q^2}{(Q^2-q^2)} - 1\right] = (2xb - b^2)\frac{q^2}{(Q^2-q^2)}$

$\frac{r_1^2}{r_2^2} = \frac{Q^2}{q^2}$

Hence $\frac{Q}{r_1} - \frac{q}{r_2} = 0$, proving that this sphere is a equipotential, with $\phi = 0$. No other equipotential surface, in

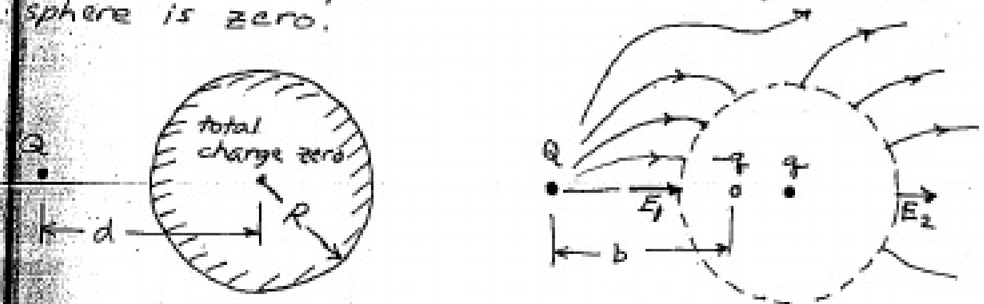
this system, is exactly spherical. We now have the solution for a point charge Q in the vicinity of a conducting sphere maintained at zero potential. That is, given a charge Q located distance d from the center of a sphere of radius R, potential zero,



we simply replace the sphere by the point charge $-q$ in the right spot. We want $x_0 = d$ and R is given, so from $R = x_0\frac{q}{Q}$ we have: $q = Q\frac{R}{d}$

and from $x_0 = \frac{bQ^2}{Q^2-q^2}$ we have: $b = d\left(1 - \frac{R^2}{d^2}\right)$

(b) To the system above we can add, without violating the boundary conditions on the field at the surface of the conducting sphere, the field of any point charge q' located at the center of the sphere. The spherical surface remains an equipotential, but with $\phi \neq 0$. If we let $q' = q$, then the net charge in the sphere is zero.



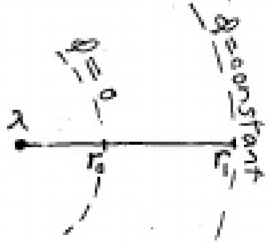
Given: $Q = 10$ esu
$d = 20$ cm $R = 10$ cm
charge zero on sphere

Equivalent system:
$q = 5$ esu $-q = -5$ esu
$b = 15$ cm

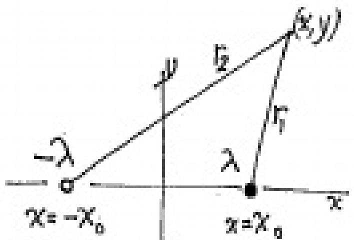
$E_1 = \frac{10}{10^2} + \frac{5}{5^2} - \frac{5}{10^2} = \frac{1}{4}$ statvolt/cm

$E_2 = \frac{10}{30^2} - \frac{5}{15^2} + \frac{5}{10^2} = \frac{7}{180}$ statvolt/cm

3.23 Consider the potential φ for a single line charge of density λ, in esu/cm. Let $\varphi = 0$ at $r = r_0$. Then since $\mathbf{E} = \frac{2\lambda}{r}\hat{r}$, the potential at a distance r_1 from the line is:



$$\varphi = -\int_{r_0}^{r_1} \frac{2\lambda}{r} dr = 2\lambda \ln \frac{r_0}{r_1}$$



With two line charges located as shown,

$$\varphi = 2\lambda \ln \frac{x_0}{r_1} + 2(-\lambda) \ln \frac{x_0}{r_2}$$

$$\varphi = 2\lambda \ln \frac{r_2}{r_1}$$

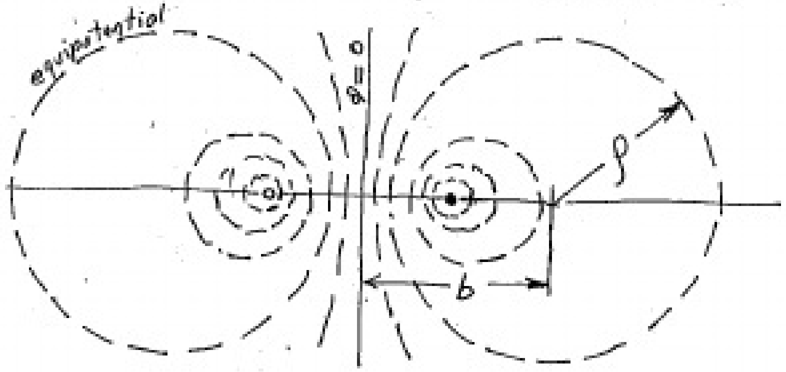
Now to show that $\frac{r_2}{r_1}$ = constant defines a circle:

Let $\frac{r_2}{r_1} = k > 1$ $\quad r_2^2 = (x+x_0)^2 + y^2 \quad r_1^2 = (x-x_0)^2 + y^2$

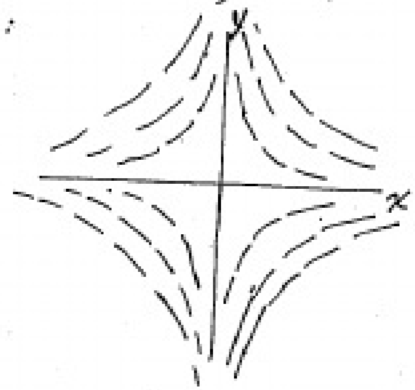
$(x+x_0)^2 + y^2 = k^2[(x-x_0)^2 + y^2]$ which reduces to

$$x^2 + y^2 - 2\frac{k^2+1}{k^2-1} x_0 x + x_0^2 = 0$$

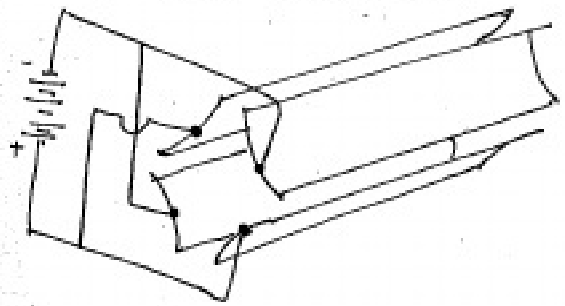
A circle of radius ρ, centered at $x = b$, $y = 0$ has the equation $x^2 + y^2 - 2bx + b^2 - \rho^2 = 0$ which is the same as the preceding equation if we take $b = \frac{k^2+1}{k^2-1} x_0$ and $\rho^2 = b^2 - x_0^2$. This all applies in the right half-plane, where $r_2/r_1 > 1$. In the left half plane $r_2/r_1 < 1$ and $\varphi < 0$.



3.24 $E_x = Ky \quad E_y = Kx$ The potential function is $\varphi = -Kxy$ (see derivation of Eq. 2.11, p. 38). Thus the equipotential surfaces are cylinders of hyperbolic cross-section:

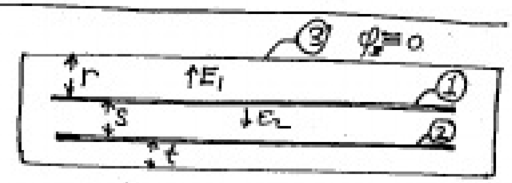


One could produce a good approximation to this field, in the central region, by four long metal strips, curved in cross-section, and alternating in potential:



3.25 $Q_1 = C_{11}\varphi_1 + C_{12}\varphi_2$

$Q_2 = C_{21}\varphi_1 + C_{22}\varphi_2$



Make $\varphi_2 = 0$ by connecting ② to ③. Then $E_1 = \varphi_1/r$ and $E_2 = \varphi_1/s$ $\quad 4\pi Q_1 = AE_1 + AE_2$ $= A\varphi_1\left(\frac{1}{r} + \frac{1}{s}\right)$ Hence $C_{11} = \frac{A}{4\pi}\left(\frac{1}{r} + \frac{1}{s}\right)$

By similar argument, $C_{22} = \frac{A}{4\pi}\left(\frac{1}{t} + \frac{1}{s}\right)$

When $\varphi_2 = 0$, $Q_2 = C_{21}\varphi_1$, and here $Q_2 = -\frac{E_2}{4\pi}A$ $= -\frac{A\varphi_1}{4\pi s}$ Thus $C_{21} = -\frac{A}{4\pi s}$. Clearly a similar argument with 1's and 2's interchanged will give $C_{12} = -\frac{A}{4\pi s}$.

3.26 C_{jk} cannot be positive. <u>Proof</u>: Suppose $\phi_k > 0$ and all other potentials are zero. At no point in the space between the conductors can the potential be < 0, for otherwise we would have a potential minimum in empty space. There-fore, at every point on the surface of every one of the conductors at zero potential, the outward potential gradient must be zero or positive. But this means that the surface charge density is zero or negative. Hence the total charge Q_j on each such conductor must be zero or negative. $Q_j = C_{jk}\phi_k$, so $C_{jk} \leq 0$.

By a similar argument, Q_k, the charge on the only conductor at positive potential, must be positive. Hence $C_{kk} > 0$, necessarily.

3.27 $Q_1 = C_{11}\phi_1 + C_{12}\phi_2 \qquad Q_2 = C_{21}\phi_1 + C_{22}\phi_2$

<u>Process (a)</u>: Step 1: $\phi_2 = 0$; ϕ_1 goes from 0 to ϕ_{1f}.

$dQ_1 = C_{11}\,d\phi_1 \qquad \text{Work} = \int_0^{\phi_{1f}} \phi_1 C_{11}\,d\phi_1 = \frac{1}{2}C_{11}\phi_{1f}^2$

Step 2: $\phi_1 = \phi_{1f}$; ϕ_2 goes from 0 to ϕ_{2f}. $\quad dQ_1 = C_{12}\,d\phi_2$

$dQ_2 = C_{22}\,d\phi_2 \qquad \text{Work} = \int_0^{\phi_{2f}} (\phi_{1f}C_{12} + \phi_2 C_{22})\,d\phi_2 = C_{12}\phi_{1f}\phi_{2f} + \frac{1}{2}C_{22}\phi_{2f}^2$

Total work done $= \frac{1}{2}C_{11}\phi_{1f}^2 + \frac{1}{2}C_{22}\phi_{2f}^2 + C_{12}\phi_{1f}\phi_{2f}$

<u>Process (b)</u> In process (b) 1's and 2's are interchanged throughout. Hence the expression for total work done must be $\frac{1}{2}C_{11}\phi_{1f}^2 + \frac{1}{2}C_{22}\phi_{2f}^2 + C_{21}\phi_{1f}\phi_{2f}$. Since the final state is the same and there is no dissipation of energy in the charging process, the work done must be the same, if energy is to be conserved. It follows that $C_{12} = C_{21}$.

3.28

$$\phi(x_0+\delta, y_0, z_0) = \phi(x_0, y_0, z_0) + \delta\frac{\partial\phi}{\partial x} + \frac{\delta^2}{2}\frac{\partial^2\phi}{\partial x^2} + \frac{\delta^3}{6}\frac{\partial^3\phi}{\partial x^3} + \cdots$$

$$\phi(x_0-\delta, y_0, z_0) = \phi(x_0, y_0, z_0) - \delta\frac{\partial\phi}{\partial x} + \frac{\delta^2}{2}\frac{\partial^2\phi}{\partial x^2} - \frac{\delta^3}{6}\frac{\partial^3\phi}{\partial x^3} + \cdots$$

$$\phi(x_0, y_0+\delta, z_0) = \phi(x_0, y_0, z_0) + \delta\frac{\partial\phi}{\partial y} + \frac{\delta^2}{2}\frac{\partial^2\phi}{\partial y^2} + \frac{\delta^3}{6}\frac{\partial^3\phi}{\partial y^3} + \cdots$$

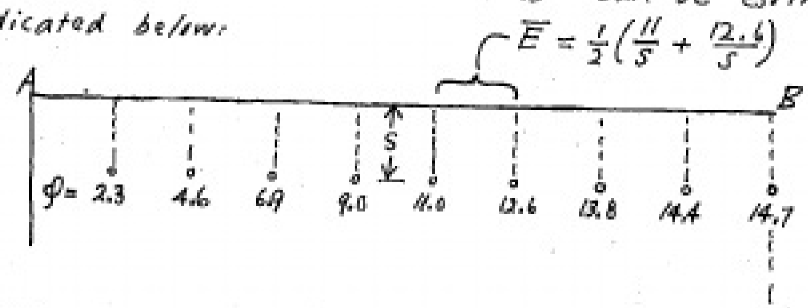
etc.

The average of the six values of ϕ is

$$\bar{\phi} = \frac{1}{6}\left[6\phi_0(x_0, y_0, z_0) + \delta^2\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right) + \delta^4(\cdots) + \cdots\right]$$

If $\nabla^2\phi = 0$, we are left with $\bar{\phi} = \phi(x_0, y_0, z_0) + \delta^4(\ldots) + \cdots$

3.30 To find the capacitance per cm length of capacitor: Take $\phi = 100$ statvolts at inner tube, $\phi = 0$ at outer tube, as in Figure on page 418. Assume side of unit grid square is s cm. The surface charge density on the inner surface of the outer tube is $\frac{1}{4\pi} \times$ electric field. The average electric field strength at the surface, in any of the intervals between A and B can be estimated as indicated below:

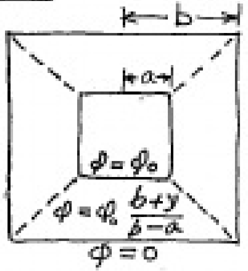


The total charge between A and B, (per cm length of conductor in direction perpendicular to page) is

$$q = \sum \frac{s}{4\pi}\cdot\bar{E} = \frac{s}{4\pi}\left[\frac{1}{2}\frac{(0+2.3)}{s} + \frac{1}{2}\frac{(2.3+4.6)}{s} + \cdots + \frac{1}{2}\frac{(14.4+14.7)}{s}\right] = \frac{1}{4\pi}\left[2.3 + 4.6 + \cdots + 14.4 + \frac{14.7}{2}\right]$$

$= \frac{81.95}{4\pi}$. The charge on the whole square is 8 times this or $\frac{163.9}{\pi}$. $\quad C = \frac{Q}{\phi} = \frac{163.9}{\pi \cdot 100} = 0.522$ This is the capacitance, in cm, per cm length of capacitor, a dimensionless number. Notice that s dropped out.

3.32



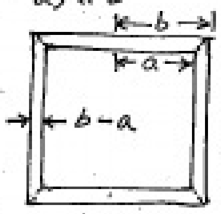
$|\nabla\phi| = \frac{\phi_0}{b-a}$ in all four sections, excluding the diagonal boundaries. These boundaries themselves take up zero volume. ϕ does not satisfy Laplace's Eq. on these boundaries, so it cannot be the exact solution. In fact $\nabla^2\phi$ is infinite on such a boundary. However $|\nabla\phi|$ is not infinite on any boundary, so in computing $\int|\nabla\phi|^2 dv$ over the whole volume, these singularities contribute nothing. Thus

$$U' = \frac{1}{8\pi}\int|\nabla\phi|^2 dv = \frac{1}{8\pi}\left(\frac{\phi_0}{b-a}\right)^2 \times \text{volume} = \frac{\phi_0^2}{8\pi}\frac{[(2b)^2-(2a)^2]}{(b-a)^2}$$

$$= \frac{\phi_0^2}{2\pi}\frac{b^2-a^2}{(b-a)^2} = \frac{\phi_0^2}{2\pi}\frac{b+a}{b-a}.$$

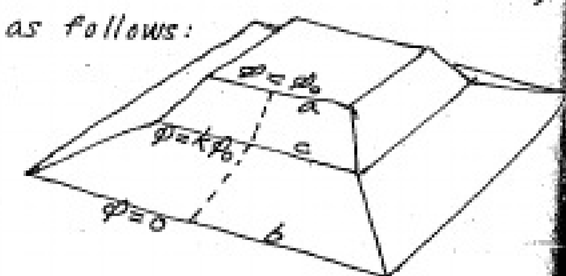
We know that U' will be greater than the U computed for the exact ϕ, and since $U = \frac{1}{2}C\phi_0^2$, $\quad \frac{1}{2}C\phi_0^2 < \frac{\phi_0^2}{2\pi}\frac{b+a}{b-a}$ or $C < \frac{1}{\pi}\frac{b+a}{b-a}$

Our approximation to C is thus greater than the true C and is: $C \approx \frac{1}{\pi}\frac{b+a}{b-a}$. For $b = 3a$, as in Problem 3.30, this gives $C \approx \frac{2}{\pi} = 0.638$. The more accurate estimate in Problem 3.30 gave $C \simeq 0.522$. Our approximate formula ought to be more accurate for $(b-a) \ll b$. In fact, in that limit it reduces to the "parallel-plate" approximation, as we should expect, for $b-a$ is the separation everywhere except at the corners and $4(b+a)$ is the average perimeter length, or plate width. That is,

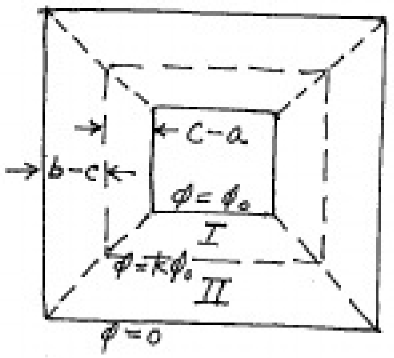


$C = \frac{1}{4\pi}\frac{\text{Area}}{\text{separation}}$ gives $C \approx \frac{1}{4\pi}\frac{4(b+a)}{b-a} = \frac{1}{\pi}\left(\frac{b+a}{b-a}\right)$.

A more flexible trial function of the same type can be constructed as follows: This ϕ has two adjustable parameters, a constant $k<1$ and a length c, the half-side of the



intermediate square.



In regions I, $|\nabla\phi| = \frac{\phi_0 - k\phi_0}{c-a}$

In regions II, $|\nabla\phi| = \frac{k\phi_0}{b-c}$

Volume of regions I $= 4(c^2-a^2)$

Volume of regions II $= 4(b^2-c^2)$

Hence:

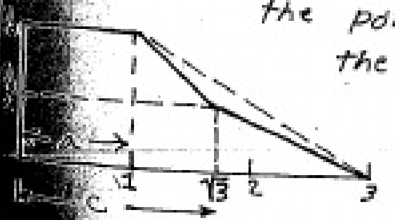
$$U' = \frac{1}{8\pi}\int|\nabla\phi|^2 dv = \frac{\phi_0^2}{2\pi}\left[(1-k)^2\frac{c+a}{c-a} + k^2\frac{b+c}{b-c}\right]$$

We have now to minimize U', for given ϕ_0, a, and b, by adjusting k and c. First we hold c constant and set $\frac{\partial U'}{\partial k} = 0$. This leads, after a little algebra, to the condition $k = \frac{(b-c)(c+a)}{2c(b-a)}$. Substituting this k gives us $U' = \phi_0^2\left[\frac{c^2 + c(b+a) + ab}{4\pi(b-a)c}\right]$

Setting now $\frac{\partial U'}{\partial c} = 0$ leads to the condition $c = \sqrt{ab}$ which makes $U' = \frac{\phi_0^2}{4\pi}\frac{\sqrt{b}+\sqrt{a}}{\sqrt{b}-\sqrt{a}}$. From $U' > \frac{1}{2}C\phi_0^2$ we get, as before, an approximate formula for the capacitance C:

$$C \approx \frac{1}{2\pi}\frac{\sqrt{b}+\sqrt{a}}{\sqrt{b}-\sqrt{a}}$$

For $b/a = 3$, as in the previous examples, this gives $C \approx \frac{1}{2\pi}\frac{2.732}{0.732} = 0.594$. This is clearly a better (i.e., lower) estimate than the figure 0.638. Notice that for this case, with $b=3$, $a=1$, $c=\sqrt{3}$ and $k=\frac{1}{2}$, the potential surface is more like the true one – it "sags" in the space between the conductors.



CHAPTER 4

4.1 Electron current density $= \underline{J}_e = \rho_e \underline{v}_e = 10^{11}(-e)\underline{v}_e$

$J_e = 10^{11}\times 4.8\times10^{-10}\times 10^{8} = 4.8\times10^{9}$ esu/cm²sec

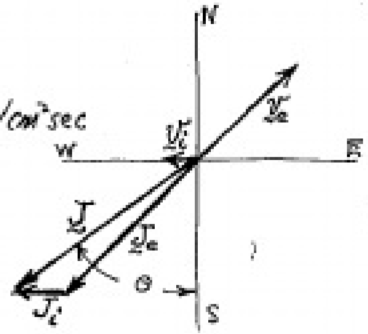
$J_i = 2\times5\times10^{10}\times4.8\times10^{-10}\times10^{7} = 4.8\times10^{8}$

$\underline{J} = \underline{J}_e + \underline{J}_i$

$J_{west} = .707\times4.8\times10^{9} + 4.8\times10^{8} = 3.87\times10^{9}$

$J_{south} = .707\times4.8\times10^{9} = 3.39\times10^{9}$ $\qquad \theta = \tan^{-1}\frac{3.87}{3.39} = 48.8°$

$J = \sqrt{J_{west}^2 + J_{south}^2} = 5.14\times10^{9}$ esu/cm²sec $= 1.71$ amperes/cm²



4.2 Each electron makes $\frac{3\times10^{8}\text{ meters/sec}}{240\text{ meters}} = 1.25\times10^{6}$ round trips per second. Charge passing any point, per second $= 1.25\times10^{6}\times10^{11}\times4.8\times10^{-10} = 6\times10^{7}$ esu/sec or .020 amp

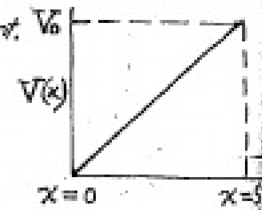
4.3



$E = 40$ statvolts/cm. Total σ on belt (both sides) $= 2\times\frac{40}{4\pi} = \frac{20}{\pi}$ esu/cm². Current carried by 30 cm wide belt with speed 2000 cm/sec:

$I = 30\times\left(\frac{20}{\pi}\right)\times2000 = 3.82\times10^{5}$ esu/sec $= 0.127$ milliamp

4.4 $V(x) = V_0\frac{x}{s}$, since E is constant.



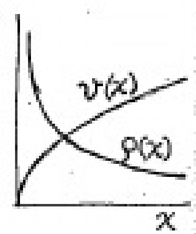
At any x, $\frac{1}{2}mv^2 = eV = eV_0\frac{x}{s}$

$v = \sqrt{\frac{2eV_0x}{ms}}$

$J = \frac{I}{A} = \rho v$

$\rho = \frac{I}{Av} = \frac{I}{A}\sqrt{\frac{ms}{2eV_0x}}$

$v \sim \sqrt{x} \qquad \rho \sim \frac{1}{\sqrt{x}}$



4.5 Silver has atomic weight 108, density 10.5 gm/cm³

No. of atoms/cm³ $= \frac{6\times10^{23}\times10.5}{108} = 6\times10^{22}$/cm³

With 1 free electron per atom $\rho_e = 6\times10^{22}e$

30 amp $= 30\times3\times10^{9} = 9\times10^{10}$ esu/sec $= I$

$J = I/A \qquad A = \frac{\pi}{4}\times10^{-2}$ cm² $\qquad J = \rho_e\bar{v}$

$\bar{v} = \frac{J}{\rho_e} = \frac{I}{A\rho_e} = \frac{9\times10^{10}}{(\frac{\pi}{4})\times10^{-2}\times6\times10^{22}\times4.8\times10^{-10}} = 0.4$ cm/sec

4.6 $I = 1.5\times10^{-6}$ amp $= 1.5\times10^{-6}\times3\times10^{9} = 4.5\times10^{3}$ esu/sec

$E = \frac{1000\text{ volts}}{2\text{ cm}} = \frac{3.33\text{ statvolts}}{2\text{ cm}} = 1.67$ statvolts/cm

$J = \frac{I}{\text{area}} = \frac{4.5\times10^{3}}{10^{2}} = 45$ esu/cm²sec

$\sigma = \frac{J}{E} = \frac{45}{1.67} = 27$ sec^{-1}

$\sigma = e^2\left(\frac{N_+\tau_+}{M_+} + \frac{N_-\tau_-}{M_-}\right)$ (Eq. 17, p. 122)

> average molecular wt. for air ≈ 29 (oxygen: 32, nitrogen: 28)

Here we assume $\tau_+ = \tau_- = \tau \qquad \tau = \frac{10^{-5}\text{ cm}}{5\times10^{4}\text{ cm/sec}} = 2\times10^{-10}$ sec

$M_+ = M_- \approx \frac{29}{6\times10^{23}} = 4.8\times10^{-23}$ gm $\qquad N_+ = N_- = N$

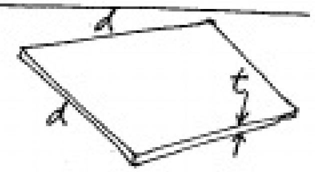
$\sigma = \frac{2e^2N\tau}{M}$ where N is the number of ions of one sign.

$N = \frac{M\sigma}{2e^2\tau} = \frac{4.8\times10^{-23}\times27}{2(4.8\times10^{-10})^2\times2\times10^{-10}} \approx 1.5\times10^{7}$ per cm³

The number of molecules/cm³ at room temperature and 1 atmosphere is $\frac{6\times10^{23}}{22400}\cdot\frac{273}{293} = 2.5\times10^{19}$

The fraction of molecules ionized (including both positive and negative ions) is $\frac{3\times10^{7}}{2.5\times10^{19}} = 1.2\times10^{-12}$.

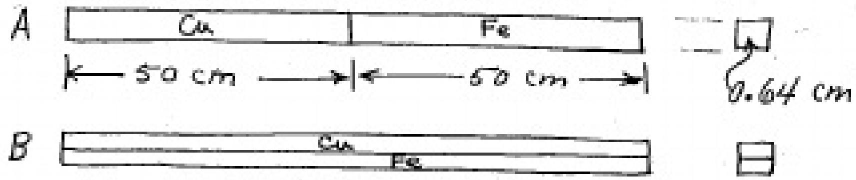
4.7 Resistance between opposite edges of square: $R = \frac{\rho d}{dt} = \rho/t$



$R = \frac{2.83\times10^{-6}\text{ ohm-cm}}{5\times10^{-7}\text{ cm}} = 5.66$ ohms

Note that this is independent of the size of the square.

4.8



$R_A = R_{Cu} + R_{Fe} = \frac{1.77\times10^{-6}\times50}{.64} + \frac{10\times10^{-6}\times50}{.64} = 9.19\times10^{-4}$ ohm

$R_B = \frac{R_{Cu}R_{Fe}}{R_{Cu}+R_{Fe}}$ where now $R_{Cu} = \frac{1.77\times10^{-6}\times100}{.32} = 5.53\times10^{-4}$

and $R_{Fe} = \frac{10\times10^{-6}\times100}{.32} = 31.25\times10^{-4}$ ohm

$R_B = \frac{5.52\times31.25}{5.52+31.25}\times10^{-4} = 4.70\times10^{-4}$ ohm

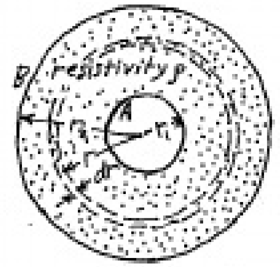
In A more power is dissipated in the iron, because I is the same in both and $P = I^2R$

In B more power is dissipated in the copper, because V is the same across both and $P = V^2/R$.

4.9 If you assume that the resistivity of the copper remains constant and its density remains constant as it is stretched, then you should expect a 0.1% increase in length to produce an increase in resistance of 0.2%. For the cross-sectional area must decrease by 0.1% to keep the volume constant, and $R \sim \frac{\rho\cdot\text{length}}{\text{area}}$. Actually the density of an elastic solid does not remain constant under stretch. In copper, a 0.1% increase in length is accompanied by a decrease in cross-section of only 0.065%. Taking that into account, we might expect a resistance increase of 0.165% rather than 0.2%. But a more important, and more interesting effect is a change in resistivity ρ, under tension. In copper this change combined with the geometrical change about doubles the expected resistance change, to 0.33% for 0.1% stretch, as shown by

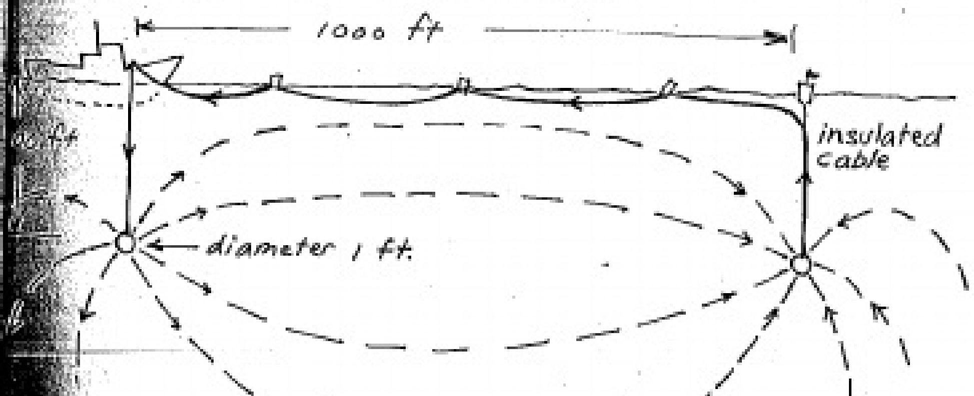
measurements of P. W. Bridgman in 1923. To explain the change in ρ one needs a complete theory of electronic conduction in crystals.

4.10 Consider first, as suggested, the resistance between the concentric spheres A and B. The current flows radially through the medium of resistivity ρ, which can be divided up into concentric spherical shells, all in series. The resistance between the inner and outer surface of a shell of radius r, thickness dr, is $\frac{\rho\,dr}{4\pi r^2}$. Then $R_{AB} = \int_{r_1}^{r_2}\frac{\rho\,dr}{4\pi r^2} = \frac{\rho}{4\pi}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$

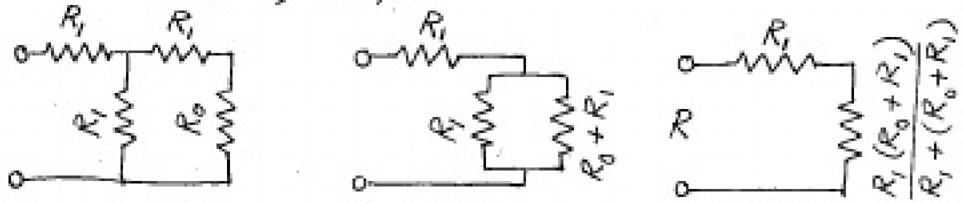


If $r_2 \gg r_1$ the second term is negligible: $R_{AB} \approx \frac{\rho}{4\pi r_1}$. This shows that the resistance is contributed chiefly by the central region surrounding the inner sphere. If the size of the outer boundary is unimportant, its shape will also be unimportant. In the ocean path in the problem, therefore, the resistance will consist almost entirely of the two "spreading resistances" in series: $R = 2\frac{\rho}{4\pi a}$. With $a = 6$ inches $= 15$ cm, and $\rho = \frac{1}{\sigma} = 25$ ohm-cm,

$R = 2\cdot\frac{25}{4\pi\times15} = 0.27$ ohm. To this must be added, of course, the resistance of the conductor in the insulated cable which completes the circuit.



4.11 To find the input resistance R, reduce the circuit step by step:

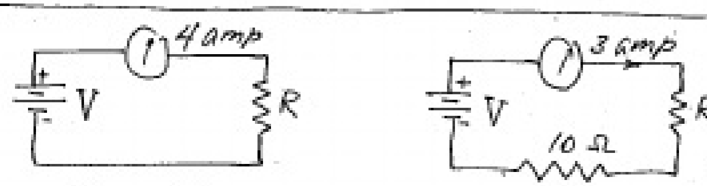


$R = R_1 + \frac{R_1(R_0+R_1)}{R_1+(R_0+R_1)}$ We want $R = R_0$, or

$R_0(2R_1+R_0) = R_1(2R_1+R_0) + R_1(R_1+R_0)$.

Collecting terms: $R_0^2 = 3R_1^2$ or $R_1 = R_0/\sqrt{3}$

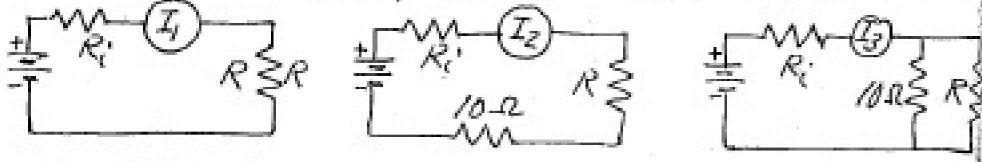
4.12



$V = 4R$ $V = 3(R+10)$

Solve for R and V: $R = 30\ \Omega$ $V = 120$ volts

If the battery had internal resistance R_i, three measurements of current would be needed to determine R_i, R, and V, for instance:



$V = I_1(R_i+R)$ $V = I_2(R_i+R+10)$ $V = I_3\left(R_i + \frac{10R}{R+10}\right)$

4.13 Let P = power dissipated in resistor R

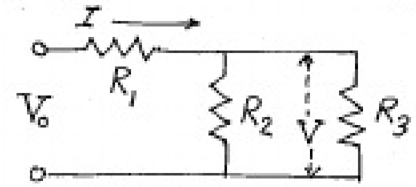


$P = I^2R$ $I = \frac{\mathcal{E}}{R+R_i}$ $P = \frac{\mathcal{E}^2 R}{(R+R_i)^2}$

$\frac{dP}{dR} = \mathcal{E}^2\frac{(R+R_i)^2 - 2R(R+R_i)}{(R+R_i)^4} = \mathcal{E}^2\frac{R_i - R}{(R+R_i)^3}$

For $R = R_i$ $\frac{dP}{dR} = 0$. Also, $\frac{dP}{dR} < 0$ for $R > R_i$, and $\frac{dP}{dR} > 0$ for $R < R_i$. Hence this is condition for maximum P.

4.14 Let P be the power dissipated in resistor R_3.



$I = \frac{V_0}{R_1 + \frac{R_2R_3}{R_2+R_3}} = \frac{V_0(R_2+R_3)}{R_1R_2+R_1R_3+R_2R_3}$

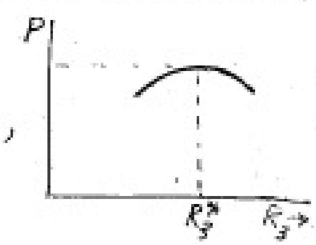
$V = V_0 - IR_1 = V_0 - \frac{V_0R_1(R_2+R_3)}{R_1R_2+R_1R_3+R_2R_3} = V_0\frac{R_2R_3}{R_1R_2+R_1R_3+R_2R_3}$

$P = \frac{V^2}{R} = V_0^2\frac{R_2^2R_3}{(R_1R_2+R_1R_3+R_2R_3)^2}$. We want $\frac{\partial P}{\partial R_3} = 0$:

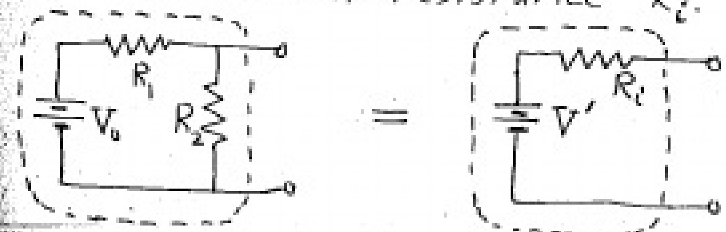
$\ldots^2(R_1R_2+R_1R_3+R_2R_3)^2 - 2R_2^2R_3(R_1+R_2)(R_1R_2+R_1R_3+R_2R_3) = 0$

his reduces to $R_1R_2 - R_2R_3 - R_1R_3 = 0$, or $R_3 = \frac{R_1R_2}{R_1+R_2}$

this condition is satisfied, P is an extremum ctually a maximum) with respect to variation of with R_1 and R_2 constant. enote by R_3^* this value of R_3. A mall change in R_3, $R_3 \to R_3^* + \Delta R_3$, ll cause only a second-order ange in P, that is, $\Delta P \sim (\Delta R_3)^2$.



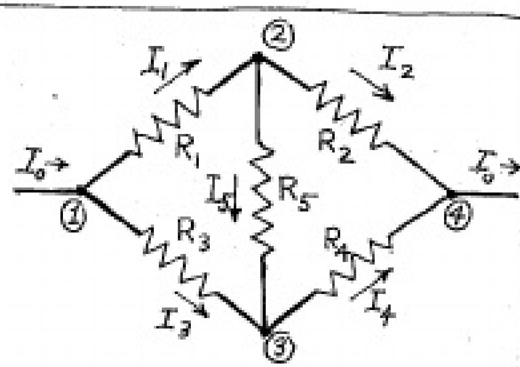
mment: Note that $\frac{R_1R_2}{R_1+R_2}$ is just the sistance of R_1 and R_2 in parallel. As "seen by" R_3 constant voltage source V_0 and the resistors and R_2 are equivalent to a battery with V' and internal resistance R_i.



open-circuit voltage of the box on the left is

$V_0 \frac{R_2}{R_1+R_2}$, that of the box on the right is V'. Let us set $V' = \frac{V_0 R_2}{R_1+R_2}$ and compare the short-circuit current on the left, $I = \frac{V_0}{R_1}$ with that on the right, $I = \frac{V'}{R_i} = \frac{V_0}{R_i}\frac{R_2}{(R_1+R_2)}$. These will be the same if $R_i = \frac{R_1R_2}{R_1+R_2}$. This suffices to establish the equivalence of the two boxes, and the condition $R_3 = \frac{R_1R_2}{R_1+R_2}$ for maximum power in R_3 is now seen as an example of the theorem of Problem 13.

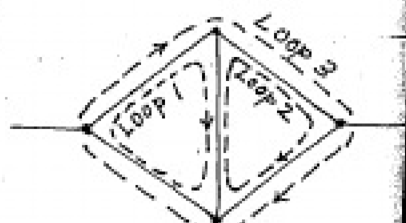
4.15



Node	Equation
①	$I_0 - I_3 - I_1 = 0$
②	$I_1 - I_2 - I_5 = 0$
③	$I_3 - I_4 + I_5 = 0$
④	$I_2 + I_4 - I_0 = 0$

The first three equations are obviously independent, for each contains a current that appears in neither of the other two. If we add the first three equations and multiply by -1, we get the fourth equation. So the number of independent equations is three.

Now, apply the loop condition:



Loop	Equation
1	$R_1I_1 + R_5I_5 - R_3I_3 = 0$
2	$R_2I_2 - R_4I_4 - R_5I_5 = 0$
3	$R_1I_1 + R_2I_2 - R_4I_4 - R_3I_3 = 0$

The third equation is the sum of the other two; only two of the loop equations are independent.

Let's take the first three node equations and the first two loop equations as our set of five:

$$\left.\begin{aligned} &I_0 - I_3 - I_1 = 0 \\ &I_1 - I_2 - I_5 = 0 \\ &I_3 - I_4 + I_5 = 0 \\ &R_1I_1 + R_5I_5 - R_3I_3 = 0 \\ &R_2I_2 - R_4I_4 - R_5I_5 = 0 \end{aligned}\right\}$$

Using the first three to eliminate I_1, I_2 and I_3:

$$I_1 = I_0 - I_3 = I_0 - I_4 + I_5$$
$$I_2 = I_1 - I_5 = I_0 - I_4$$
$$I_3 = I_4 - I_5$$

$$\left.\begin{aligned} &R_1(I_0 - I_4 + I_5) - R_3(I_4 - I_5) + R_5I_5 = 0 \\ &R_2(I_0 - I_4) - R_4I_4 - R_5I_5 = 0 \end{aligned}\right\}$$

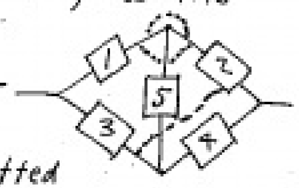
Collecting terms:
$$\left\{\begin{aligned} &(-R_1 - R_3)I_4 + (R_1+R_3+R_5)I_5 + R_1I_0 = 0 \\ &(-R_2 - R_4)I_4 - R_5I_5 + R_2I_0 = 0 \end{aligned}\right.$$

Solving for I_4 and I_5:

$$I_5 = I_0(R_2R_3 - R_1R_4)/D$$
$$I_4 = I_0(R_1R_2 + R_2R_3 + R_2R_5 + R_1R_5)/D$$

where $D = R_1R_2 + R_3R_4 + R_1R_4 + R_2R_3 + R_1R_5 + R_2R_5 + R_3R_5 + R_4R_5$
$= (R_1+R_3)(R_2+R_4) + R_5(R_1+R_2+R_3+R_4)$

The best check on our algebra, at this point, is the symmetry of the expressions for D and for I_5. In fact, given the expression for I_4, we should be able to guess the formulas for I_1, I_2 and I_3 without any algebra. Notice that the combinations appearing in the numerator of I_4 are those indicated by the dotted connections in the diagram at the right.



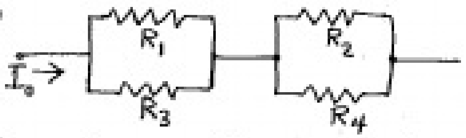
Then we must expect for each of the other side-arm currents:

$$I_3 = I_0(R_1R_2 + R_1R_5 + R_2R_5 + R_1R_4)/D$$
$$I_2 = I_0(R_3R_4 + R_3R_5 + R_4R_5 + R_1R_4)/D$$
$$I_1 = I_0(R_3R_4 + R_3R_5 + R_4R_5 + R_2R_3)/D$$

In fact this is what we get by completing the algebraic solution.

In the special case $R_5 = 0$, the circuit

reduces to



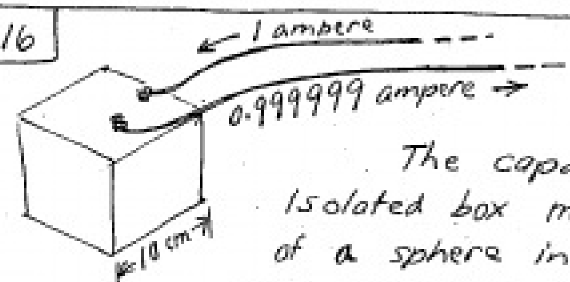
In this circuit $I_1 R_1 = I_3 R_3$ from which we find at once $I_1 = I_0 \frac{R_3}{R_1+R_3}$. That is just what our formula gives, if we set $R_5 = 0$.

The most interesting result is the formula for I_5, the current in the cross-arm:

$$I_5 = \frac{I_0 (R_2 R_3 - R_1 R_4)}{(R_1+R_3)(R_2+R_4) + R_5 (R_1+R_2+R_3+R_4)}$$

If $\frac{R_1}{R_2} = \frac{R_3}{R_4}$, I_5 will be zero. The bridge is said to be balanced. The Wheatstone bridge is simply this circuit with a galvanometer in the position of R_5. The balance condition is very easy to derive, if that is all we are interested in. Our formula for I_5 tells us much more; we can predict the galvanometer current in the <u>unbalanced</u> condition, and thus predict the sensitivity of the bridge as an instrument.

4.16



The capacitance of the isolated box must be roughly that of a sphere intermediate between the inscribed and the circumscribed sphere – say 6 cm radius. $C = 6$ cm $\approx 6 \times 10^{-12}$ farads. This object is acquiring positive charge at the rate of 10^{-6} amperes or 10^{-6} coulomb/sec.

$\frac{dV}{dt} = \frac{1}{C}\frac{dQ}{dt} = \frac{1}{6\times 10^{-12}} \times 10^{-6} = \frac{1}{6} \times 10^{6}$ volts/sec. Hence the potential will rise by 1000 volts in 6 milliseconds.

4.17 Eq. 34, p. 139: $I = -\frac{dQ}{dt} = \frac{V_0}{R} e^{-t/RC}$

At any instant after $t=0$, the power dissipated in the resistor is $P = I^2 R = \frac{V_0^2}{R} e^{-2t/RC}$

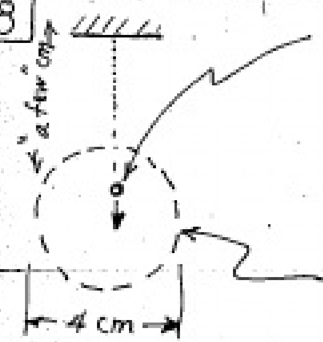
The total energy dissipated is $\int_0^\infty P\,dt = \int_0^\infty \frac{V_0^2}{R} e^{-\frac{2t}{RC}} dt$

$$= \frac{V_0^2}{R} \cdot \frac{RC}{2} \int_0^\infty e^{-x} dx = \frac{1}{2} C V_0^2$$

Suppose we have a 1 microfarad capacitor charged to 100 volts. $Q = CV = 10^{-4}$ coulombs. One electron charge is 1.6×10^{-19} coulombs. If $t_0 = RC$ is the time constant, we shall have one electron left when $e^{-t/t_0} = \frac{1.6\times 10^{-19}}{10^{-4}}$, or $\frac{-t}{t_0} = \ln 1.6 \times 10^{-15}$

This gives $t = t_0 \times 2.3 \log_{10}(6 \times 10^{14}) = 34\, t_0$. So if the time constant were ≈ 1 sec, we'd be down to "one electron" in a little over half a minute!

4.18



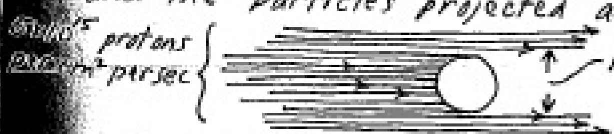
graphite particle, roughly spherical
diameter 10^{-3} cm $= d$
volume $= \frac{\pi}{6} d^3 \approx 0.5 \times 10^{-9}$ cm^3
mass = volume × density $\approx 10^{-9}$ gm
(taking density of graphite ≈ 2 gm/cm^3)

outline of proton beam cross-section
area $= 4\pi$ cm^2

Beam current $= 10^{-2}$ amp $= 3\times 10^{9} \times 10^{-2} = 3 \times 10^{7}$ esu/sec

Flux of protons, per cm^2 per sec $= \frac{3\times 10^{7}}{4\pi \times 4.8 \times 10^{-10}} = 5 \times 10^{15}$/cm^2-sec

When the particle first enters the beam, it will intercept protons at a rate determined by the proton flux and the particle's projected area $\frac{\pi d^2}{4}$:



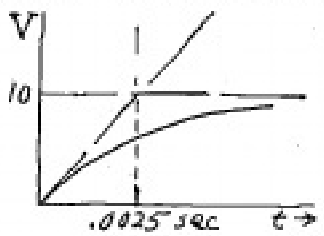
$5\times10^{15}\times\frac{\pi}{4}\times10^{-6} = 4\times10^{9}$ protons/sec.

The initial rate of increase of charge Q on the particle is therefore $\frac{dQ}{dt} = 4\times10^{9}\times4.8\times10^{-10} \approx 2$ esu/sec

This will cause the potential of the particle to rise at a rate $\frac{dV}{dt} = \frac{1}{C}\frac{dQ}{dt}$. C = capacitance of the particle = 0.5×10^{-3} cm (same as its radius)

$$\frac{dV}{dt} = \frac{1}{.5\times10^{-3}}\cdot 2 = 4\times10^{3} \text{ statvolts/sec}$$

When the potential has reached 3000 volts, or 10 statvolts, no more protons can strike it. If charging proceeded at the initial rate of 4×10^{3} statvolts/sec this condition would be attained in $\frac{10}{4\times10^{3}} = .0025$ sec. However, as the particle acquires charge it will repel some of the protons that would otherwise hit it, and the rate of charge collection will decrease. The potential will approach its limiting value, 10 statvolts, asymptotically. But .0025 sec



is considerably less than the time required for the particle to fall through the beam; for after falling, say, 5 cm a body has velocity $\sqrt{2gh} = \sqrt{2\cdot 980\cdot 5} = 100$ cm/sec. It will travel only 0.25 cm in the time 0.0025 sec. Conclusion: the particle's potential V will be close to its limiting value, 10 statvolts, during most of its passage through the beam.

The energy deposited in the particle while it is intercepting protons can be found as follows: Let V_0 = 10 statvolts = 3000 volts. $V_0\,dQ$ is the initial kinetic energy of a group of protons with total charge dQ. When these come to rest in a body at potential V, their potential energy is $V\,dQ$, and hence

he energy that has been dissipated is $(V_0-V)\,dQ$. ince $V = \frac{Q}{C}$ at any time, the total energy issipated in the course of the charging process s:

$$\int_0^{CV_0}(V_0-V)\,dQ = \int_0^{CV_0}\left(V_0-\frac{Q}{C}\right)dQ = \frac{1}{2}CV_0^{2}$$

otice that this is just equal to the final stored energy.) ith $V_0 = 10$ statvolts and $C = 0.5\times10^{-3}$ cm we get: nergy dissipated in particle $= \frac{1}{2}\times.5\times10^{-3}\times10^{2} = .025$ erg.

The specific heat of almost anything is of order f magnitude 1 joule/gm-deg or 10^{7} ergs/gm-deg. Our article has a mass 10^{-9} gm. Dumping .025 erg n such a particle can cause its temperature to se, at most, by only a few degrees. Conclusion: e particle does not get hot.

The horizontal force on the particle must be equal o the rate of loss of momentum by the protons. et's compute the initial momentum of a 3 kv oton: $\frac{1}{2}mv_0^{2} = eV_0$, $m^{2}v_0^{2} = 2eV_0m$, so

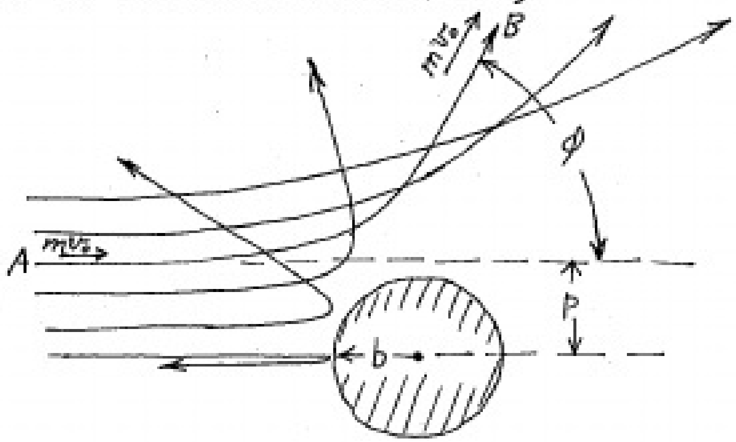
$$v_0 = \sqrt{2eV_0m} = \sqrt{2\times4.8\times10^{-10}\times10\times1.6\times10^{-24}} = 1.24\times10^{-16} \text{ gm-cm/sec}$$

en the particle is uncharged, it intercepts 4×10^{9} otons/sec, stopping them dead. The resulting force is $F = 4\times10^{9}\times1.25\times10^{-16} = 5\times10^{-7}$ dynes

horizontal acceleration of the particle then: $a = F/M = 5\times10^{-7}/10^{-9} = 500$ cm/sec^{2} about half the acceration of gravity.

After the particle acquires its full charge CV_0, o longer catches protons but it does deflect reducing their forward momentum. So is still a force on the particle. Is it or smaller than the initial force com- above? One might guess that it would be of me order of magnitude, but in fact it is derably greater. The situation is now

similar to Rutherford scattering



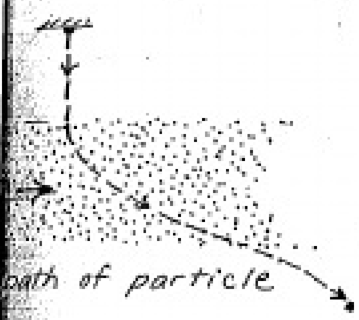
The distance of closest approach, head-on, is b, which is here the radius of the graphite particle. A particle which would have passed a distance p from the center, if the sphere were not charged, is deflected along the trajectory AB. Rutherford showed that $\cot(\frac{\phi}{2}) = 2p/b$ (see Vol. I p. 464). The loss in forward momentum is $mv_0 - mv_0 \cos\phi = 2mv_0 \sin^2\frac{\phi}{2} = 2mv_0 \frac{b^2}{b^2+4p^2}$. To find the total rate of momentum transfer to the particle we have to integrate over the whole cross-section of the beam. If the beam radius is R (2 cm in our case) and the proton flux is n protons/cm³ in the beam, the force on the scattering center is

$$\int_0^R \left(2mv_0 \frac{b^2}{b^2+4p^2}\right)(2\pi n p\, dp) = \frac{\pi}{2} m n v_0 b^2 \ln\left(1+4\frac{R^2}{b^2}\right)$$

This differs from the initial force on the uncharged particle by the factor $\frac{1}{2}\ln(1+4\frac{R^2}{b^2})$. In fact for a beam of constant density and unlimited cross-section, the force appears to grow infinite. This kind of divergence is characteristic of pure coulomb scattering. In our case, with $R=2$

nd $b = 0.5\times10^{-3}$, the factor $\frac{1}{2}\ln(1+\frac{4R^2}{b^2})$ is $\ln(6.4\times10^7)\approx 9$. The force on the fully charged particle s approximately 9 times the force on the uncharged article: $F = 9\times5\times10^{-7}$ dynes $= 4.5\times10^{-6}$ dynes

he corresponding horizontal acceleration is 4.5×10^3 cm/sec² cting for .04 sec, roughly the time required for he particle to fall vertically through the beam, this cceleration leads to a horizontal velocity of 180 cm/sec.



Clearly if the beam were directed upward, it could support such a particle — even a somewhat larger particle — since 4.5×10^3 cm/sec² is several times the acceleration of gravity.

4.21 From Eq. 15, p. 121, $\bar{u} = \frac{Ee\tau}{M}$ where τ he mean time between collisions for an ion of mass M, arge e. (t was used instead of τ in Eq. 15). If mobility is defined by $\bar{u} = E\mu$, then our theory predicts

$$\mu = \frac{e\tau}{M}$$

gas at constant temperature, $\tau \sim 1/\text{density}$. e we expect μ to vary as 1/density.

$\mu_{He} = 5.1$ cm²/volt-sec and $\mu_e = 500$ cm²/volt-sec, equivalent values in CGS units are:

$\mu_{He} = 300\times5.1 = 1530$ cm²/statvolt-sec

$\mu_e = 300\times500 = 15\times10^4$ cm²/statvolt-sec

$M_{He} = \frac{4}{6\times10^{23}} = 6.7\times10^{-24}$ gm $m_e = 9\times10^{-28}$ gm

$\tau_{He} = \frac{\mu M}{e} = \frac{1530\times6.7\times10^{-24}}{4.8\times10^{-10}} = 2.1\times10^{-11}$ sec.

$\tau_e = \frac{\mu m}{e} = \frac{15\times10^4\times9\times10^{-28}}{4.8\times10^{-10}} = 2.8\times10^{-13}$ sec.

ions and the electrons are in thermal equilibrium

in respect to their velocities, at the same temperature,

$$\bar{v}_{He}/\bar{v}_e = (m/M)^{\frac{1}{2}}$$

[This follows from the equipartition of kinetic energy,

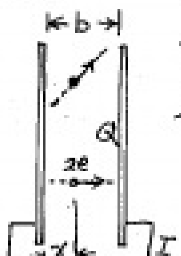
$$\tfrac{1}{2}M\overline{v_{He}^2} = \tfrac{1}{2}m\overline{v_e^2} \;]$$

If L is the mean free path, $L = \bar{v}\tau$. Thus

$$\frac{L_{He}}{L_e} = \frac{\tau_{He}}{\tau_e}\sqrt{\frac{m}{M}} = \frac{2.1\times10^{-11}}{2.8\times10^{-13}}\sqrt{\frac{9\times10^{-28}}{6.7\times10^{-24}}} = 0.8$$

The mean free paths are about equal. Only about 1% of the conductivity is contributed by the positive ions, since $\mu_{He}/\mu_e \simeq .01$ in this gas.

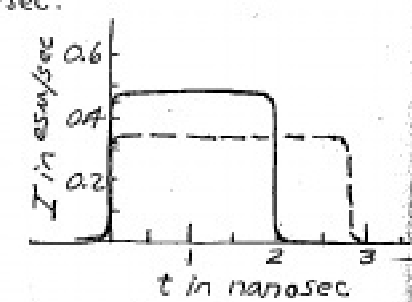
4.23 $Q = -2e\dfrac{x}{b}$ $I = -\dfrac{dQ}{dt} = \dfrac{2e}{b}\dfrac{dx}{dt} = \dfrac{2e}{b}v$



$v = 10^8$ cm/sec $b = 0.2$ cm

$$I = \frac{2\times4.8\times10^{-10}\times10^8}{0.2} = 0.48 \text{ esu/sec}$$

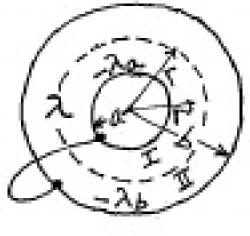
The current pulse lasts for $\dfrac{0.2\text{ cm}}{10^8\text{ cm/sec}}$ $= 2\times10^{-9}$ sec $= 2$ nanosec.



If the path slopes at 45°;

$\dfrac{dx}{dt} = 10$ cos 45°. The current pulse is correspondingly reduced in amplitude and stretched out in time.

To treat the cylindrical electrode case, we have first to solve the equivalent of Problem 3.17 for that geometry. We can use the same trick, smearing out the charge Q, which is at radius r onto a cylindrical surface of that radius. Suppose this results in



a density of charge λ per unit length of cylinder. The induced charge on the inner conductor, similarly smeared out, is $-\lambda_a$, that on the outer is $-\lambda_b$. $\lambda_b + \lambda_a = \lambda$. In region I the field is $E = -\dfrac{2\lambda_a}{r'}$ at any radius r' and the rise in potential from $r'=a$ to $r'=r$ is $2\lambda_a \ln\dfrac{r}{a}$. In region II the field is $E = \dfrac{2(\lambda-\lambda_a)}{r'}$ and the rise in potential coming in from $r'=b$ to $r'=r$ is $2(\lambda-\lambda_a)\ln\dfrac{b}{r}$. Since the inner and outer electrodes are at the same potential,

$$2\lambda_a \ln\frac{r}{a} = 2(\lambda-\lambda_a)\ln\frac{b}{r}$$

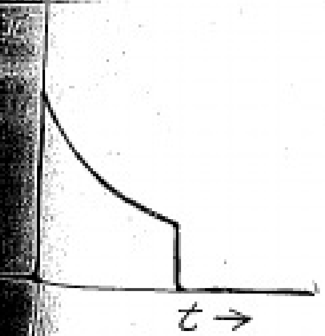
Solving for λ_a:

$$\frac{\lambda_a}{\lambda} = \frac{\ln b - \ln r}{\ln b - \ln a}$$

This must be the same as $\dfrac{Q_a}{Q}$, where Q_a is the induced charge on the inner electrode when Q is at radius r.

If an α-particle moves from a to b at constant radial speed, we shall have

$$I = \frac{dQ_a}{dt} = \frac{dQ_a}{dr}\frac{dr}{dt} = v_r\frac{d}{dr}Q\left[\frac{\ln b - \ln r}{\ln b - \ln a}\right]$$

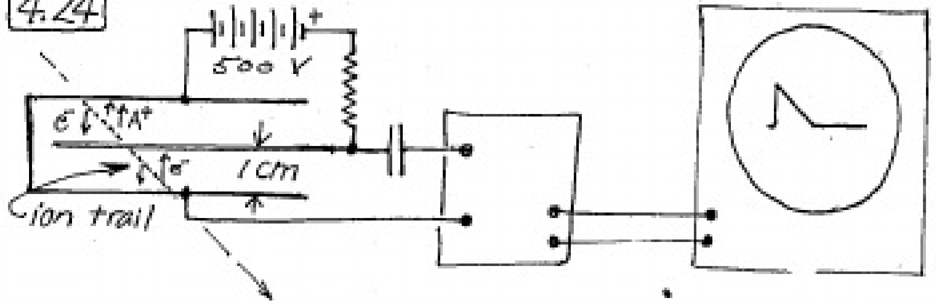


$$= \frac{Q v_r}{(\ln b - \ln a)\,r}$$

$$= \frac{Q v_r}{(\ln b - \ln a)(a + v_r t)}$$

$$= \frac{2e v_r}{(a + v_r t)\ln\left(\frac{b}{a}\right)}$$

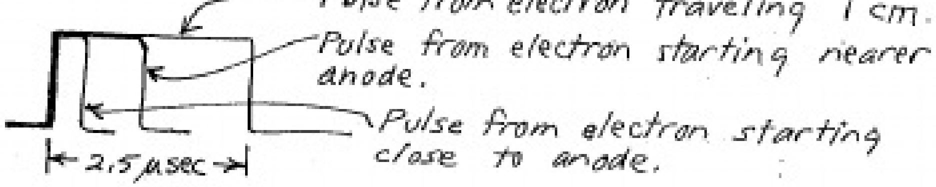
4.24



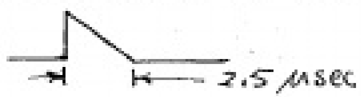
$\mu_{A^+} = 1.6$ cm²/volt-sec $\qquad \mu_e = 800$ cm²/volt-sec

at $E = 500$ volts/cm, time to cross 1 cm gap is $\frac{1}{500\mu}$. Electrons take $\frac{1}{500 \times 800}$ sec $= 2.5$ microsec.

Electrons starting closer to anode move at same speed, take less time.



In the event shown, the starting positions are uniformly distributed between anode and cathode. We therefore expect the superposition of electron pulses to have the form.



The positive ion pulse should be similar in shape, but its amplitude will be 500 times less, its duration 500 times longer. It will be a current pulse of the same sign. If we could observe all the details, in the idealized case the current pulse would look like this:



4.25

$$\left.\begin{array}{l} \frac{d^2V}{dx^2} = -4\pi\rho \\ \rho v = J = -\frac{I}{A} \\ \frac{1}{2}mv^2 = eV \end{array}\right\}$$

$$v = \left(\frac{2eV}{m}\right)^{\frac{1}{2}}$$

$$\rho = -\frac{I}{Av} = -\frac{I}{A}\left(\frac{m}{2eV}\right)^{\frac{1}{2}}$$

$$\frac{d^2V}{dx^2} = \frac{4\pi I}{A}\left(\frac{m}{2e}\right)^{\frac{1}{2}} V^{-\frac{1}{2}}$$

$$K \equiv \frac{4\pi I}{A}\left(\frac{m}{2e}\right)^{\frac{1}{2}}$$

$$\frac{d^2V}{dx^2} = K V^{-\frac{1}{2}}$$

Multiply by $2\frac{dV}{dx}$:

$$2\frac{dV}{dx}\frac{d^2V}{dx^2} = 2KV^{-\frac{1}{2}}\frac{dV}{dx}$$

Integrate:

$$\left(\frac{dV}{dx}\right)^2 = 4KV^{\frac{1}{2}} + C_1$$

We require $\frac{dV}{dx} = 0$ and $V = 0$ for $x = 0$. Hence $C_1 = 0$.

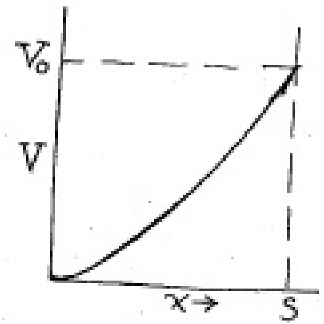
$$\frac{dV}{dx} = 2K^{\frac{1}{2}}V^{\frac{1}{4}}, \quad \text{or} \quad \frac{dV}{V^{1/4}} = 2K^{\frac{1}{2}}dx$$

Integrate: $\frac{4}{3}V^{\frac{3}{4}} = 2K^{\frac{1}{2}}x + C_2 \qquad C_2 = 0$

$$V^{\frac{3}{2}} = \frac{9K}{4}x^2 = \frac{9\pi I}{A}\left(\frac{m}{2e}\right)^{\frac{1}{2}} x^2$$

$V = V_0$ for $x = s$

$$I = \frac{A}{9\pi s^2}\left(\frac{2e}{m}\right)^{\frac{1}{2}} V_0^{\frac{3}{2}}$$



For given s and A, I is completely determined by V_0. Notice that $V(x) \sim x^{\frac{4}{3}}$.

With V_0 given, $V(x) = V_0\left(\frac{x}{s}\right)^{\frac{4}{3}}$.

4.26 The metal ring which has been rotating with speed v_0 is suddenly stopped. The conduction electrons must eventually also come to rest, in the sense that their average velocity, initially v_0, is reduced to zero. The average force required, on each electron, is $m\frac{dv}{dt}$. It arises from the relative motion of electron and lattice. When a steady current flows in a stationary conductor under the influence of an electric field E, which exerts a force eE on each electron, the "lattice friction" is equal and opposite to eE. Let us make the forces equal in the two cases by setting $eE = m\frac{dv}{dt}$. Then the relative motion of electrons and lattice will be the same. But we know that E causes a current density $E\sigma = J$. Therefore the relative motion of electrons and lattice in our ring must, at any instant, be equivalent to a current density



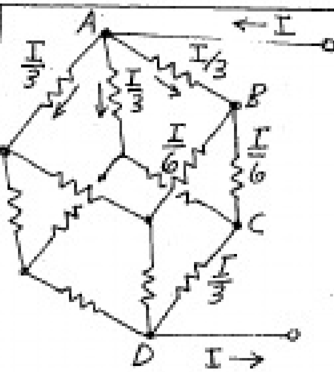
$$J = \sigma E = \sigma \frac{m}{e}\frac{dv}{dt} \qquad I = AJ$$

and $Q = \int I\,dt$. Thus the total charge transport occurring during the deceleration is

$$Q = A\int \frac{\sigma m}{e}\frac{dv}{dt}dt = \frac{A\sigma m}{e}\int_{v_0}^{0} dv = (-)\frac{A\sigma m v_0}{e}$$

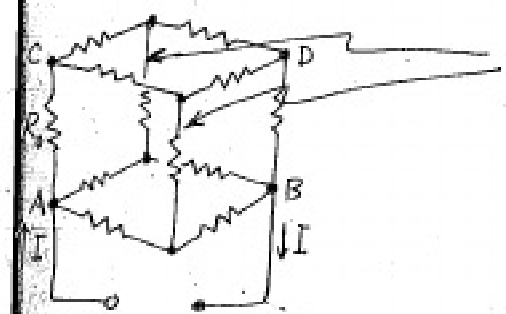
[We haven't kept track of signs, and will drop the (–).] If the carriers are in fact free, negative, charged particles, the current pulse will appear as a positive current in the sense opposite to the rotation of the ring.

4.27

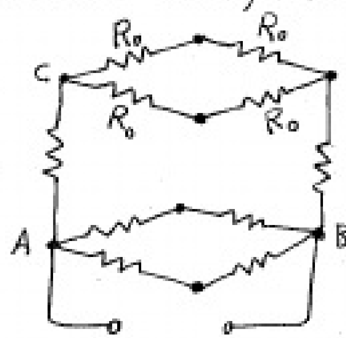


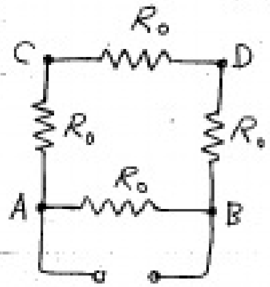
All resistors have the same value, R_0. Symmetry requires the current division shown. Proceeding along any path such as ABCD, we find the potential drop from A to D: $V_{AD} = \frac{I}{3}R_0 + \frac{I}{6}R_0 + \frac{I}{3}R_0 = \frac{5}{6}R_0 I$

Then $R_{AD} = V_{AD}/I = \frac{5}{6}R_0$



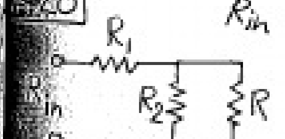
Current must be zero in these branches, by symmetry. Remove them, leaving:





$$R_{AB} = \frac{3R_0^2}{3R_0+R_1} = \frac{3}{4}R_0$$

4.28



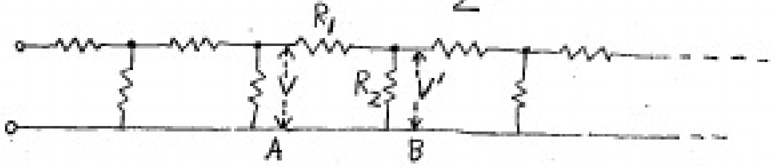
$R_{in} = R_1 + \frac{R_2 R}{R_2+R}$. Setting $R_{in} = R$, we have $R = R_1 + \frac{R_2 R}{R_2 + R}$

$$R^2 - R_1 R - R_1 R_2 = 0$$

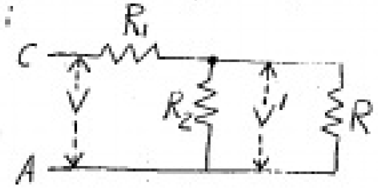
$$R = \frac{R_1 \pm \sqrt{R_1^2 + 4R_1R_2}}{2}$$

The negative root is not ordinarily realizable. We are

left with: $R = \dfrac{\sqrt{R_1^2 + 4R_1R_2} + R_1}{2}$



Given V, what is V'? Replace the ladder to the right of B by R. That won't affect anything to the left of B. We now have:



The resistance between C and A is just R_1 as we know. Evidently, then,

$$\frac{V'}{V} = \frac{\left(\frac{RR_2}{R+R_2}\right)}{R} = \frac{R_2}{R+R_2}$$

If we want $\frac{V'}{V} = \frac{1}{2}$, we must have $R = R_2$

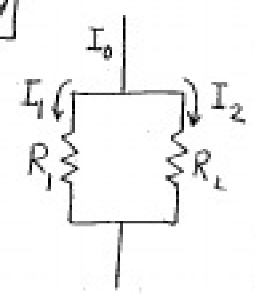
Then $2R_2 = \sqrt{R_1^2 + 4R_1R_2} + R_1$

$(2R_2 - R_1)^2 = R_1^2 + 4R_1R_2 \qquad 4R_2^2 = 8R_1R_2$

or $R_2 = 2R_1$

To terminate the ladder after any section, simply connect on a single resistor of value R.

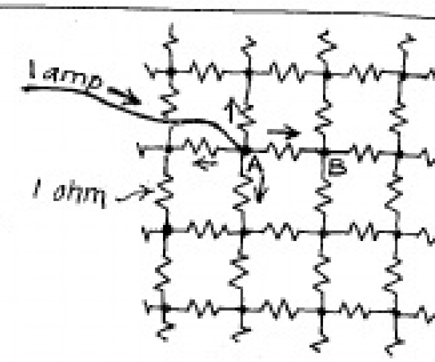
4.29



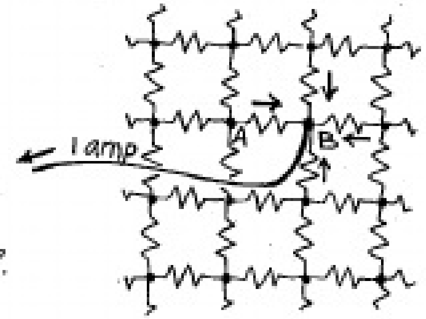
$$P = I_1^2R_1 + I_2^2R_2$$
$$= I_1^2R_1 + (I_0 - I_1)^2R_2$$
$$= I_1^2(R_1+R_2) - 2I_0R_2I_1 + R_2I_0^2$$
$$\frac{dP}{dI_1} = 2I_1(R_1+R_2) - 2I_0R_2$$

To minimize P, set $\frac{dP}{dI_1} = 0$. This gives the condition $I_1 = I_0\frac{R_2}{R_1+R_2}$, which we also get from Ohm's law. Observe that P is minimum, not maximum, for $dP/dI_1 < 0$ if I_1 is less than $I_0R_2/(R_1+R_2)$

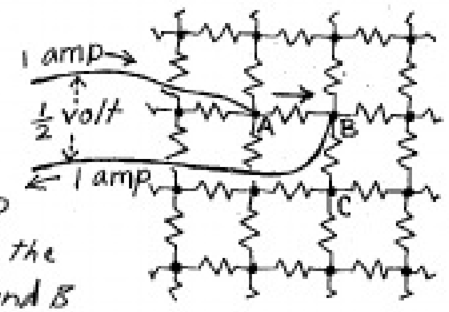
4.30 Let one ampere flow into net at A. Return lead (not shown) is connected all around rim at infinity. By symmetry, current in resistor AB is $\frac{1}{4}$ amp, flowing from A to B.



Let one ampere flow out of net at B. Return lead connected around rim at infinity. By symmetry current in resistor AB is $\frac{1}{4}$ amp, flowing from A to B.

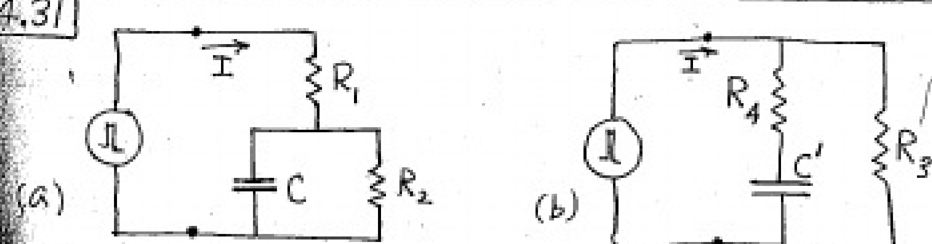


Superpose the two states above. Current at infinity is zero. Current in AB is $\frac{1}{2}$ amp. Since AB is a one-ohm resistor, the potential drop across AB is $\frac{1}{2}$ volt. Hence the network with terminals at A and B is equivalent to a $\frac{1}{2}$ ohm resistance.



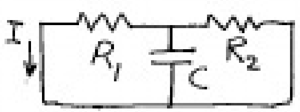
Note: the resistance between A and C cannot be found by any such easy trick. The answer is known, however; it is $\frac{2}{\pi}$ ohms!

4.31

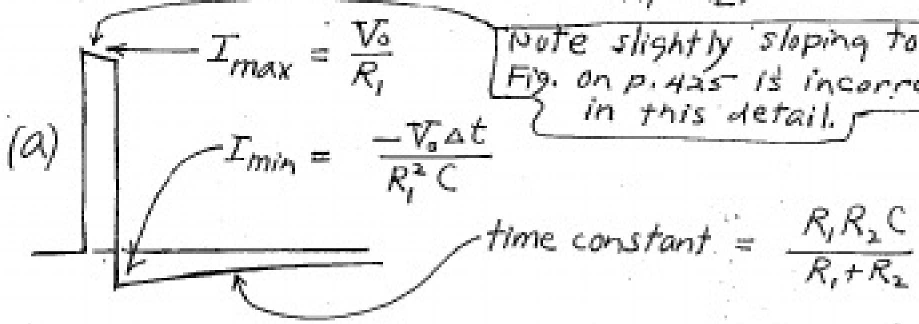


The generator produces a very short emf pulse, amplitude V_0 and duration Δt. In circuit (a)

the capacitor C is initially uncharged; the potential drop across C is zero, and increases only slowly as C acquires charge. At $t = 0$, $I = V_0/R_1$, decreasing only slightly in the short time Δt. Practically all this current goes though C, not through R_2. [Our assumption about Δt is, specifically: $\Delta t \ll R_1 C$ and $\Delta t \ll R_2 C$.] At time $t = \Delta t$, C has acquired a charge $Q = I\Delta t \approx \frac{V_0}{R_1}\Delta t$. The potential drop across C is then $\frac{Q}{C} = \frac{V_0 \Delta t}{R_1 C}$. With the disappearance of the emf, C discharges through R_1 and R_2 in parallel:



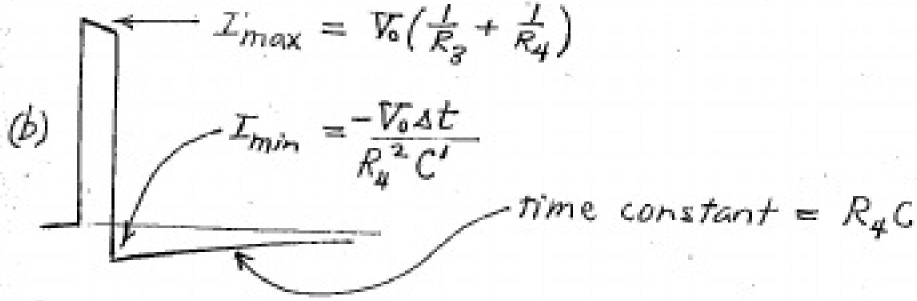
The current flowing backward through the input circuit is $\frac{1}{R_1}\left(\frac{V_0 \Delta t}{R_1 C}\right)$, decaying exponentially with the time constant $\left(\frac{R_1 R_2}{R_1 + R_2}\right)C$.



Note slightly sloping top; Fig. on p. 425 is incorrect in this detail.

In circuit (b) the initial current, with C' uncharged is $V_0\left(\frac{1}{R_3} + \frac{1}{R_4}\right)$, of which the part V_0/R_4 flows into C'. In time Δt C' acquires charge $Q = \frac{V_0 \Delta t}{R_4}$. The potential drop across C' is $\frac{V_0 \Delta t}{R_4 C'}$.

Thereafter, C' discharges through R_4 and the input circuit only, with initial current $\frac{V_0 \Delta t}{R_4^2 C'}$ and time constant $R_4 C'$.



The responses (a) and (b) will be alike if:

$$\frac{1}{R_1} = \frac{1}{R_3} + \frac{1}{R_4} \qquad \text{(same } I_{max}\text{)}$$

$$\frac{1}{R_1^2 C} = \frac{1}{R_4^2 C'} \qquad \text{(same } I_{min}\text{)}$$

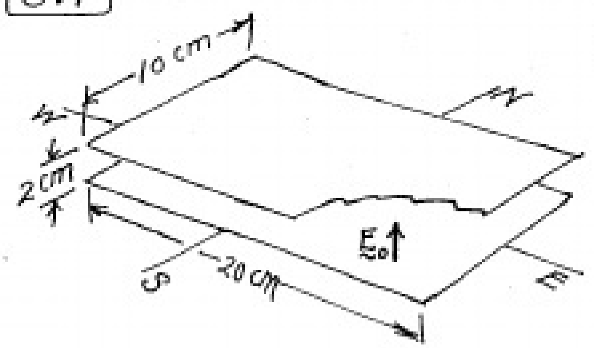
$$\frac{R_1 R_2 C}{R_1 + R_2} = R_4 C' \qquad \text{(same decay time constant)}$$

Solving the last two equations for R_4 and C', and then the first for R_3 we get:

$$R_4 = \frac{R_1(R_1 + R_2)}{R_2} \qquad C' = C\frac{R_2^2}{(R_1 + R_2)^2} \qquad R_3 = R_1 + R_2$$

CHAPTER 5

5.1



In rest frame of plates:

$$E_0 = \frac{1 \text{ statvolt}}{2 \text{ cm}} = 0.5 \text{ statvolts/cm}$$

$$\sigma_0 = \frac{E_0}{4\pi}$$

$$Q_0 = \sigma_0 A_0$$

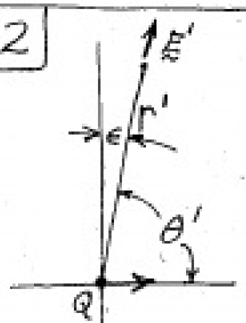
Number of excess electrons on negative plate $= \frac{Q_0}{e}$

$$= \frac{E_0 A_0}{4\pi e} = \frac{0.5 \times 10 \times 20}{4\pi \times 4.8 \times 10^{-10}} = 1.66 \times 10^{10} \text{ electrons}$$

Frame F_1 is moving east at 0.6 c. In F_1, the plates are moving west at 0.6 c. Their east-west dimension is $20\sqrt{1-.6^2} = 16$ cm. NS dimension is unchanged, as is the vertical separation. The number of electrons on the negative plate is the same, 1.66×10^{10}. But σ_1 and E_1 are greater, by the factor $\gamma = \frac{1}{\sqrt{1-.6^2}} = 1.25$ $\quad E_1 = \gamma E_0 = 0.625 \frac{\text{statvolts}}{\text{cm}}$

Frame F_2 is moving upward at 0.6 c. In F_2, the plate dimensions are 20×10 cm, the vertical separation is $2\sqrt{1-.6^2} = 1.6$ cm. The number of electrons is unchanged. $\sigma_2 = \sigma_0$ and $E_2 = E_0 = 0.5$ statvolts/cm

5.2



$$E' = \frac{Q}{r'^2} \frac{(1-\beta^2)}{(1-\beta^2 \sin^2\theta')^{3/2}}$$

$$1 - \beta^2 = \frac{1}{\gamma^2}$$

$$\sin^2\theta' = 1 - \cos^2\theta' \approx 1 - \epsilon^2, \text{ for } \epsilon \ll 1$$

$$(1-\beta^2\sin^2\theta')^{3/2} \approx \left[1 - \left(1 - \frac{1}{\gamma^2}\right)(1-\epsilon^2)\right]^{3/2}$$

$$= \left[\frac{1}{\gamma^2} + \epsilon^2 - \frac{\epsilon^2}{\gamma^2}\right]^{3/2} \approx \frac{1}{\gamma^3}(1+\gamma^2\epsilon^2)^{3/2}, \text{ since } \gamma \gg 1.$$

Thus $\quad E' \approx \frac{Q}{r'^2} \frac{(1/\gamma^2)}{\frac{1}{\gamma^3}(1+\gamma^2\epsilon^2)^{3/2}} = \frac{Q}{r_1'^2} \frac{\gamma}{(1+\gamma^2\epsilon^2)^{3/2}}$

In a band around the sphere between ϵ and $\epsilon + d\epsilon$ the flux is $\approx E' \times 2\pi r' \times r' d\epsilon = 2\pi Q \frac{\gamma\, d\epsilon}{(1+\gamma^2\epsilon^2)^{3/2}}$

If we wish to include half the total flux between $\epsilon = -\frac{\alpha}{2}$ and $\epsilon = \frac{\alpha}{2}$, the value of α must be such that

$$\int_{-\frac{\alpha}{2}}^{\frac{\alpha}{2}} 2\pi Q \frac{\gamma\, d\epsilon}{(1+\gamma^2\epsilon^2)^{3/2}} = \frac{1}{2}(4\pi Q) \quad \text{or} \quad \int_{-\frac{\alpha}{2}}^{\frac{\alpha}{2}} \frac{\gamma\, d\epsilon}{(1+\gamma^2\epsilon^2)^{3/2}} = 1$$

Put $x = \gamma\epsilon$:

$$\int_{-\frac{\gamma\alpha}{2}}^{\frac{\gamma\alpha}{2}} \frac{dx}{(1+x^2)^{3/2}} = \left.\frac{x}{(1+x^2)^{1/2}}\right]_{-\frac{\gamma\alpha}{2}}^{\frac{\gamma\alpha}{2}} = \frac{\gamma\alpha}{\left(1+\frac{\alpha^2\gamma^2}{4}\right)^{1/2}}$$

Setting $\frac{\gamma\alpha}{\sqrt{1+\frac{\alpha^2\gamma^2}{4}}} = 1$ gives $\alpha = \frac{2}{\gamma\sqrt{3}}$

5.3



$\gamma = 31$ (actually 33 if we use the more accurate rest mass, .938 Bev) Since $E_\perp' = \gamma E_\perp$,

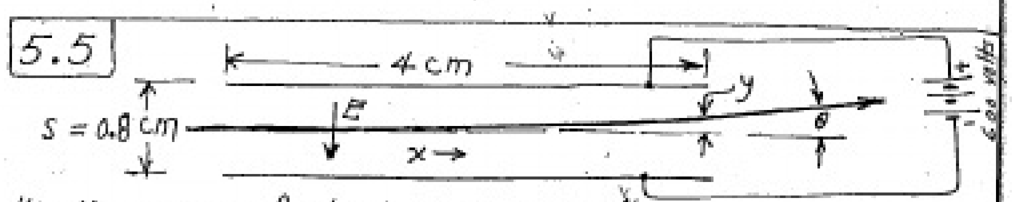
$$E_{max} = \frac{\gamma e}{r^2} \approx \frac{31 \times 4.8 \times 10^{-10}}{10^{-14}} = 1.5 \times 10^6 \frac{\text{statvolts}}{\text{cm}}$$

"duration" $\approx \frac{\alpha r}{v}$, with $v \approx c$ and α taken from Problem 5.2, $\alpha = \frac{2}{\sqrt{3}}\frac{1}{\gamma}$. Since "duration" is poorly defined anyway, we can just as well use $\frac{1}{\gamma}$ for α:

"duration" $\approx \frac{r}{\gamma c} = \frac{10^{-7}}{31 \times 3 \times 10^{10}} \approx 10^{-19}$ sec.

For 30 Bev electrons γ is about 2000 times greater: $\gamma \approx 60000$. Then $E_{max} \approx 3 \times 10^9 \frac{\text{statvolts}}{\text{cm}}$ and "duration" $\approx 5 \times 10^{-23}$ sec.

5.5



Kinetic energy of electron is 250 kev.

Rest energy + kinetic energy = (500+250) kev = γ (500 kev)

$\gamma = 750/500 = 1.5 \qquad \beta^2 = 1 - \frac{1}{\gamma^2} = 0.555$

$\beta = 0.745 \qquad p_x = \gamma\beta mc = 1.118\, mc$

$t = \frac{4\text{ cm}}{\beta c} = \frac{4}{0.745\times 3\times 10^{10}} = 1.79\times 10^{-10}$ sec

$p_y = Eet \qquad \frac{p_y}{mc} = \frac{Eet}{mc} = \frac{Ve}{s}\frac{tc}{mc^2}$

Ve has dimensions of energy, as does mc^2; so both can be in kev: $Ve/mc^2 = 6/500$

$\frac{p_y}{mc} = \frac{6\times 1.79\times 10^{-10}\times 3\times 10^{10}}{500\times 0.8} = 0.0805 = \gamma\frac{v_y}{c}$

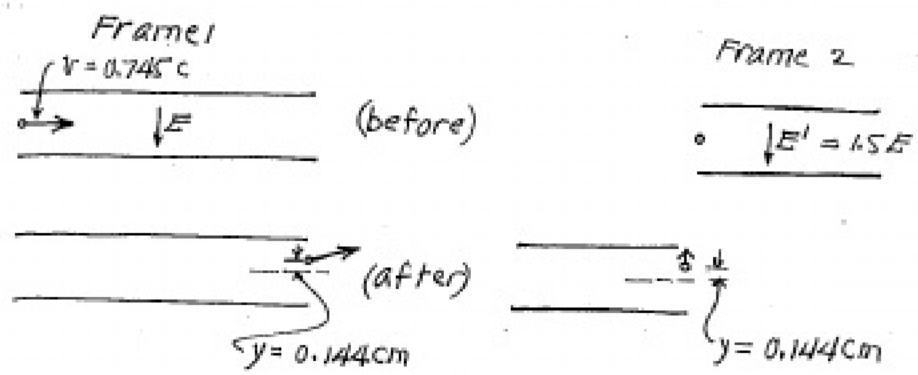
$v_y = \frac{0.0805\, c}{\gamma} = 1.61\times 10^{9}$ cm/sec (at exit)

average transverse velocity $\bar{v}_y = 0.805\times 10^{9}$ cm/sec

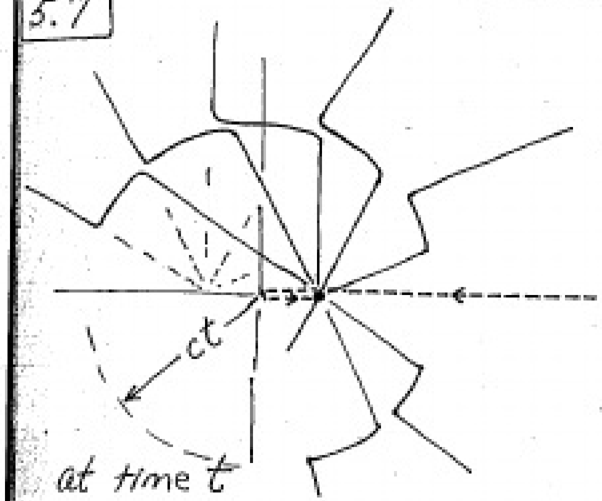
$y = \bar{v}_y t = 0.805\times 10^{9}\times 1.79\times 10^{-10} = 0.144$ cm at exit

$\theta = p_y/p_x = \frac{0.0805}{1.118} = 0.072$ radian $= 4.13°$.

In a frame in which the electron is initially at rest, the plates are moving to the left with speed $\beta c = 0.745c$. The length of the plates is $4/\gamma$ cm $= 2.67$ cm. The plates are above and below the electron for a time $t' = \frac{2.67\text{ cm}}{0.745\, c}$, during which the electron is accelerated upward in the field $E' = \gamma E = \gamma V/s$. The upward momentum acquired is the same as in the other frame: $E'et' = Eet = 0.0805\, mc$. In this frame the electron is non-relativistic, to a good approximation. $v_{y'} = .0805\, c \quad \bar{v}_{y'} = .0402\, c$ and $y' = (.0402\, c)\, t' = (.0402c)\times 2.67/(0.745\, c) = 0.144$ cm

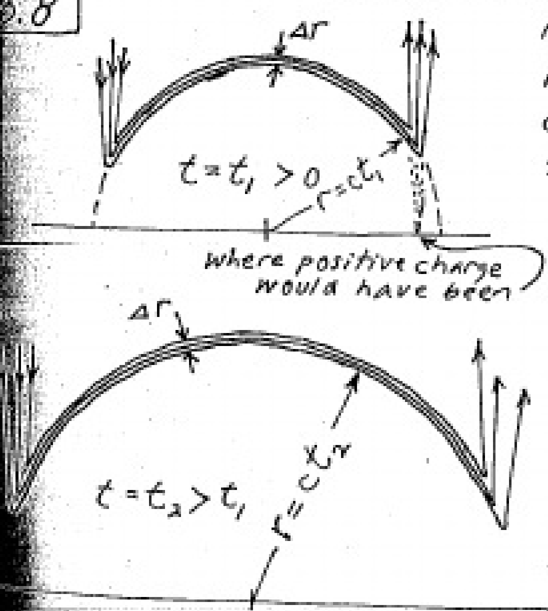


5.7



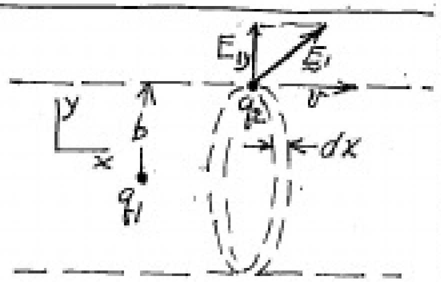
Outside a sphere of radius ct, centered on the origin, the field at time t is that of a uniformly moving charge which would be, at this time, at $x = -vt$.

5.8



No charge remains. Incoming field lines connect to outgoing field lines. Thickness Δr of shell containing connecting field is determined by duration of deceleration period. Δr remains constant. As sphere expands, E within this shell $\sim \frac{1}{r}$. This radiation is called Bremsstrahlung.

5.9 Let E_{1y} be the y-component of the electric field of q_1 at the location of q_2. Let p_{2y} be the y-component of momentum of the particle q_2.



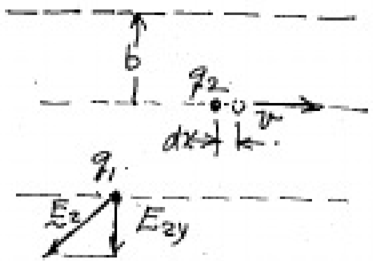
$$\frac{dp_{2y}}{dt} = E_{1y}\, q_2 \qquad dt = \frac{dx}{v} \qquad p_{2y} = 0 \ \text{ at } x = -\infty$$

for $x = +\infty$; $p_{2y} = \int_{-\infty}^{\infty} E_{1y}\, q_2\, dt = \frac{q_2}{v}\int_{-\infty}^{\infty} E_{1y}\, dx$

The surface integral of E_1 over the infinitely long cylinder of radius b is $4\pi q_1$, by Gauss's law, and this integral is just $\int_{-\infty}^{\infty} E_{1y}\cdot 2\pi b\, dx$. Hence

$$\int_{-\infty}^{\infty} E_{1y}\, dx = \frac{4\pi q_1}{2\pi b} = \frac{2q_1}{b}, \text{ from which, } p_{2y} = \frac{2q_1q_2}{vb}$$

To calculate in the same way the momentum acquired by q_1, consider a cylinder drawn with the path of q_2 as its axis. At any instant, the flux through the cylinder being $4\pi q_2$, the integral $\int_{-\infty}^{\infty} E_{2y}\, dx = \frac{2q_2}{b}$. Now q_1 is stationary; it remains at a fixed x. But clearly $\int_{-\infty}^{\infty} E_{2y}\, dt$, computed at the fixed point x, is equal to $\frac{1}{v}\int_{-\infty}^{\infty} E_{2y}\, dx$ if q_2 slides along at the constant speed v. Hence $p_{1y} = -\frac{2q_1q_2}{vb}$.

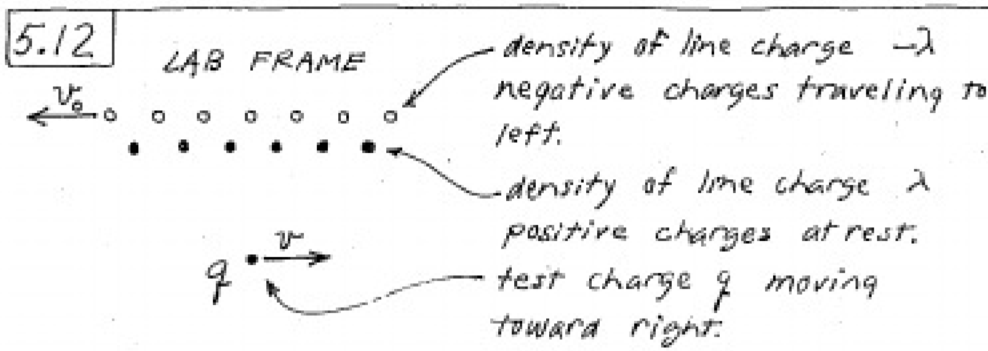


Note that, in general, $E_1 \neq E_2$, at any given time, for the field of the moving charge is not the same as the field of the charge at rest. The sum of the y-momenta of the two particles does not remain constant *during* the encounter. In fact when q_2 is passing q_1 some momentum is being exchanged with the electromagnetic field. However, when the encounter is over and q_2 is far off to the right, the net change in the momentum of one particle must be equal and opposite to that of the other, in agreement with our result.

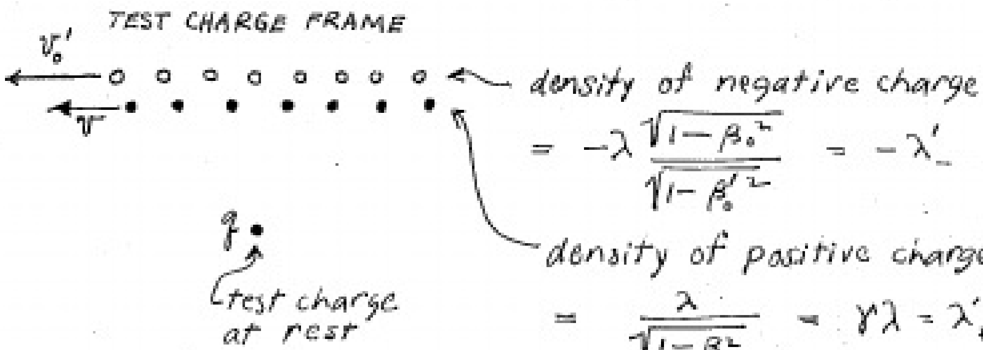
5.10 Suppose the charge is created at the origin at $t = 0$ and consider its field. There are two possibilities: (a) The field springs into existence everywhere at $t = 0$; (b) the field "grows outward" from the charge, at a finite velocity. Clearly, (a) violates relativity for it amounts to signaling with infinite speed. On hypothesis (b), Gauss's law is violated, for on a large enough sphere, at $t = t_1 > 0$, $E = 0$ all over the sphere although the sphere, at that instant contains charge.

5.11 For a proton (rest mass ≈ 1 Bev) with 1 Mev of kinetic energy, $\gamma \approx 1 + \frac{10^6}{10^9} = 1.001$. For an electron with 10 ev kinetic energy, $\gamma \approx 1 + \frac{10}{5\times10^5}$ = 1.00002. On the hypothesis that $q = \gamma q_0$, the change in electron charge would be ...gligible compared to the change in proton charge. ... H_2 is exactly neutral, we should expect He to ...w an excess positive charge, per atom, of ... order of magnitude 10^{-3}e. One liter of He ... NTP contains $6\times10^{23}\times(1/22.4) \approx 3\times10^{22}$ atoms. ... excess charge would be $\approx 10^{-3}\times3\times10^{22}$ e $\approx 10^{10}$ esu ... the electric field strength, at 10 cm distance, say, $\approx 10^{8}\ \frac{\text{statvolt}}{\text{cm}}$ ... enormous field!)

5.12



linear density of negative charge in rest frame of the negative charges: $-\lambda\sqrt{1-v_0^2/c^2} \equiv -\lambda\sqrt{1-\beta_0^2}$



density of negative charge $= -\lambda\dfrac{\sqrt{1-\beta_0^2}}{\sqrt{1-\beta_0'^2}} = -\lambda'_-$

density of positive charge $= \dfrac{\lambda}{\sqrt{1-\beta^2}} = \gamma\lambda = \lambda'_+$

$\beta = \dfrac{v}{c} \qquad \beta_0 = \dfrac{v_0}{c} \qquad \beta_0' = \dfrac{v_0'}{c}$

$$\beta_0' = \frac{\beta_0+\beta}{1+\beta_0\beta} \; ; \quad \frac{1}{\sqrt{1-\beta_0'^2}} = \frac{1}{\sqrt{1-\dfrac{(\beta_0+\beta)^2}{(1+\beta_0\beta)^2}}}$$

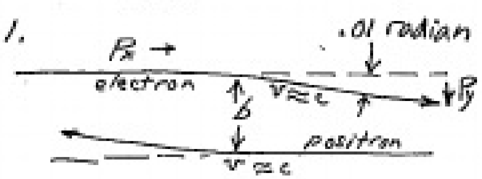
$$= \frac{1+\beta_0\beta}{\sqrt{(1-\beta^2)(1-\beta_0^2)}} .$$

The net line charge density in the frame moving with the test charge is therefore:

$$\lambda'_+ - \lambda'_- = \gamma\lambda - \lambda\frac{\sqrt{1-\beta_0^2}\,(1+\beta_0\beta)}{\sqrt{(1-\beta^2)}\sqrt{1-\beta_0^2}} = -\gamma\lambda\beta_0\beta .$$

This is the same as Eq. 41, except for the factor 2 which is absent here because the positive charges are not contributing to the current in the lab frame. From here on, the argument would duplicate that on page 176.

5.13 For a 2 Bev electron, $\gamma \approx \dfrac{2\,\text{Bev}}{.5\,\text{Mev}} = 4000$, and β is almost exactly 1.



$p_x = \beta\gamma mc = 4000\,mc$

The transverse momentum acquired was such that $\dfrac{p_y}{p_x} = 0.01$, or $p_y = 40\,mc$.

According to the result of Problem 5.9, the transverse momentum acquired by a particle of charge q_1 when another particle of charge q_2 passes it with velocity v is $\dfrac{2q_1q_2}{vb}$. To apply this, go to a frame in which the electron is at rest. The positron passes it with velocity $v \approx c$. (The positron's velocity in the lab frame was practically c; in the electron rest frame it is even closer to c.) In this frame, $p_y' = \dfrac{2q_1q_2}{cb} = \dfrac{2e^2}{cb}$. Now a transverse momentum component such as p_y, like a transverse component of any four-vector, is unchanged in a Lorentz transformation. Hence $p_y' = p_y$, so that we can set $\dfrac{2e^2}{cb} = 40\,mc$, from which,

$b = \dfrac{2e^2}{40\,mc^2} = 1.5\times10^{-14}$ cm. This distance is so small that the result can't be taken very seriously. We don't expect an electron and a positron that close together to behave exactly like classical point charges.

[Note: One might think we could have applied the formula $p_y = \dfrac{2q_1q_2}{vb}$ in the lab frame. Since the relative velocity v in the lab frame is $2c$, that would give a different answer. In the lab frame, however, each particle produces a magnetic field, which, if taken into account, contributes equally to the momentum transfer, leading to the result above: $p_y = \dfrac{2e^2}{cb}$.]

5.14 [This problem belongs in the Chapter 6 list.] The force per cm length was 10 dynes/cm, for the wires 5 cm apart, and a current of 1.5×10^{11} esu/sec in each. Between line charges λ the force per unit length is $\frac{2\lambda^2}{r}$. To get 10 dynes/cm, then, we need $\frac{2\lambda^2}{r} = 10$ or $\lambda^2 = \frac{5 \times 10}{2} = 25$ $\quad \lambda = 5$ esu/cm

The electric field strength at the surface of a 1 mm diameter wire, with this linear charge density, is $E = \frac{2\lambda}{a} = \frac{2 \times 5}{0.05} = 200$ statvolts/cm. The magnetic field due to the current, at the same radius, is $B = \frac{2I}{ac} = \frac{2 \times 1.5 \times 10^{11}}{0.05 \times 3 \times 10^{10}} \approx 200$ gauss. The numerical equality is of course not accidental. If $\frac{2\lambda^2}{r} = \frac{2I^2}{rc^2}$, then $\lambda = \frac{I}{c}$ and $\frac{2\lambda}{a} = \frac{2I}{ac}$.

5.15 $\quad I = c\sum_k n_k q_k \beta_k \qquad \lambda = \sum_k n_k q_k$

In the new frame, $\beta_k' = \frac{\beta_k + \beta}{1 + \beta_k \beta} \qquad n_k' = n_k \frac{\gamma_k'}{\gamma_k}$

$$\gamma_k' = \frac{1}{\sqrt{1 - \left(\frac{\beta_k + \beta}{1 + \beta_k \beta}\right)^2}} = \frac{1 + \beta_k\beta}{\sqrt{1 + 2\beta_k\beta + \beta_k^2\beta^2 - \beta_k^2 - 2\beta_k\beta + \beta^2}}$$

$$= \frac{1 + \beta_k\beta}{\sqrt{(1-\beta^2)(1-\beta_k^2)}} = \gamma\gamma_k(1 + \beta_k\beta)$$

$$I' = c\sum_k n_k' q_k \beta_k' = c\sum_k q_k \frac{n_k}{\gamma_k}\gamma\gamma_k \frac{(1+\beta_k\beta)(\beta_k+\beta)}{(1+\beta_k\beta)}$$

$$= c\sum_k \gamma q_k n_k \beta_k + c\beta\sum_k \gamma q_k n_k$$

$$= \gamma\left[I + c\beta\lambda\right]$$

$$\lambda' = \sum_k n_k' q_k = \sum_k q_k n_k \frac{1}{\gamma_k}\gamma\gamma_k(1 + \beta_k\beta)$$

$$= \sum_k \gamma q_k n_k + \beta\sum_k \gamma n_k q_k \beta_k$$

$$= \gamma\left[\lambda + \frac{\beta I}{c}\right]$$

CHAPTER 6

6.1 The field, in gauss, at one wire, due to the current I, in amperes, in the other wire, is: $B = \frac{2I}{10r}$. The force per unit length is $\frac{I}{10}B = \frac{2I^2}{100r} = \frac{2\times 30^2}{500} = 3.6$ dynes/cm

6.2 $F = \frac{I}{10} \times B \times \text{length} \times \text{no. of turns}$ (I in amperes)

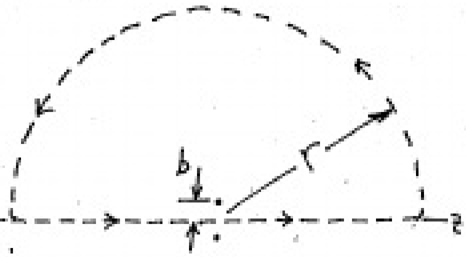
$F = 60.5 \times 980$ dynes

$$B = \frac{60.5 \times 980}{(0.5/10)\times 8 \times 15} = 9880 \text{ gauss}$$

6.3 $B_z = \frac{2\pi b^2 I}{c(b^2+z^2)^{3/2}}$ $\qquad \int_{-\infty}^{\infty} B_z\,dz = \frac{2\pi b^2 I}{c}\int_{-\infty}^{\infty}\frac{dz}{(b^2+z^2)^{3/2}}$

$$= \frac{2\pi b^2 I}{c}\left[\frac{z}{b^2(b^2+z^2)^{1/2}}\right]_{-\infty}^{\infty} = \frac{2\pi b^2 I}{c}\cdot\frac{2}{b^2} = \frac{4\pi I}{c}$$

To see why the return path can be ignored in the limit $z \to \infty$, consider the finite path out to $z = r$, returning by way of the large semicircle.



On the axis $B_z \sim \frac{1}{z^3}$, for $z \gg b$ and we may infer that, going out in any direction from the ring, $|B| \sim \frac{1}{r^3}$ as $r \to \infty$. Since the length of the semicircle is proportional to r, the integral $\int \mathbf{B}\cdot d\mathbf{s}$ over the semicircle must vanish at least as fast as $1/r^2$ as $r \to \infty$. [In fact, it vanishes just that fast.]

6.4 $B = \underbrace{\frac{1}{2}\left(\frac{2I}{rc}\right) + \frac{1}{2}\left(\frac{2I}{rc}\right)}_{\text{Each straight section contributes half the field of an infinite wire}} + \underbrace{\frac{1}{2}\left(\frac{2\pi I}{rc}\right)}_{\text{Half the field of a complete ring}}$

$$B = (2+\pi)\frac{I}{rc} = 5.1416\frac{I}{rc}$$

6.5 If $v = \frac{e^2}{\hbar}$, the ratio $\frac{v}{c} = \frac{e^2}{\hbar c}$. This ratio, frequently met in atomic physics has the value:

$$\frac{e^2}{\hbar c} = \frac{(4.803\times 10^{-10})^2}{\left(\frac{6.626\times 10^{-27}}{2\pi}\right)\times 2.998\times 10^{10}} = 0.007297, \text{ or approximately } \frac{1}{137}.$$

thus v is relatively small compared to c: $v = \frac{c}{137} = 2.19\times 10^{8}\frac{\text{cm}}{\text{sec}}$

The number of revolutions per second is $\frac{v}{2\pi a_0}$

The current $I = e\frac{v}{2\pi a_0} = \frac{4.8\times 10^{-10}\times 2.19\times 10^{8}}{2\pi \times 0.53\times 10^{-8}} = 3.16\times 10^{6}\frac{\text{esu}}{\text{sec}}$

In amperes, this is $\frac{3.16\times 10^6}{3\times 10^9} = 1.05\times 10^{-3}$ amp

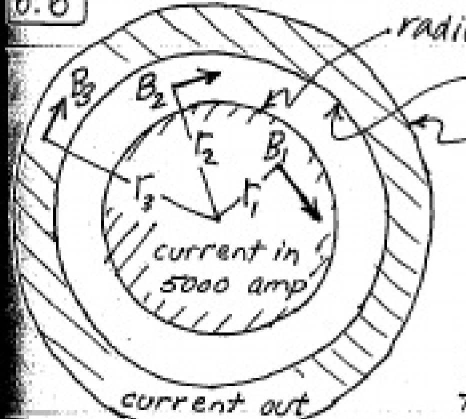
The magnetic field at the center of the orbit is that of the equivalent current ring:

$$B = \frac{2\pi I}{rc} = \frac{2\pi \times 3.16\times 10^6}{0.53\times 10^{-8}\times 3\times 10^{10}} = 1.25\times 10^{5} \text{ gauss}$$

Note: the electric field at the center of the orbit, due to the electron, is $E = \frac{e}{a_0^2}$. The magnetic field, as we have just seen, is $B = e\left(\frac{v}{2\pi a_0}\right)\cdot\frac{2\pi}{a_0 c} = \frac{e}{a_0^2}\cdot\frac{v}{c}$

That $\frac{B}{E} = \frac{v}{c}$ agrees with our result for the field of a moving charge, even though the charge is here moving on a circular path, not in a straight line at constant speed.

6.6



Given the axial symmetry, the field at any radius r will be $\frac{2I}{rc}$ where I is the current in esu/sec inside the circle of radius r. If

I is in amperes, $B = \frac{2}{10}\frac{I}{r}$.

Current inside $r_1 = 5000 \times \frac{\pi r_1^2}{\pi (2.5)^2}$ $\quad [0 \le r_1 \le 2.5]$

$$B_1 = \frac{2}{10 r_1}\,\frac{5000\, r_1^2}{(2.5)^2} = 160\, r_1 \text{ gauss}$$

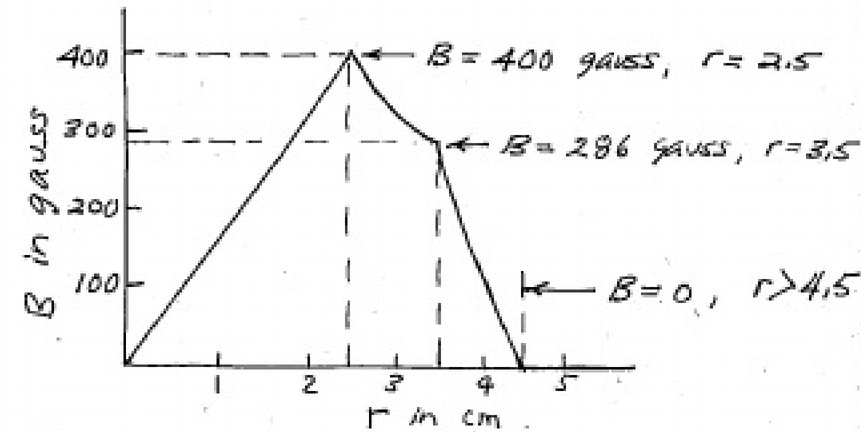
current inside $r_2 = 5000$ $\quad [2.5 \le r_2 \le 3.5]$.

$$B_2 = \frac{2}{10 r_2}\cdot 5000 = \frac{1000}{r_2} \text{ gauss}$$

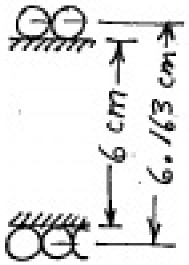
Current inside $r_3 = 5000 - 5000\,\frac{\pi(r_3^2 - 3.5^2)}{\pi(4.5^2 - 3.5^2)}$

$= 625(20.25 - r_3^2)$ $\quad [3.5 \le r_3 \le 4.5]$

$$B_3 = \frac{2}{10 r_3}\cdot 625(20.25 - r_3^2) = 125\left(\frac{20.25}{r_3} - r_3\right)$$



6.7



Length of wire per turn $= \pi(6 + .163) = 19.35$ cm

Total length $= 19.35 \times 150$ cm $= 29.0$ mete

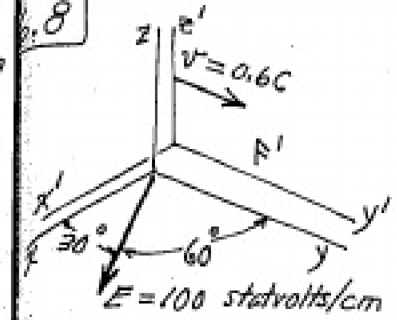
$R = 0.29$ ohms $\quad I = \frac{24}{.29} = 82.8$ amp

$B = \frac{4\pi}{10}(\text{amp-turns/cm}) = \frac{4\pi}{10}\times 5 \times 82.8 = 520$ ga

Power $= 82.8 \times 24 \approx 2000$ watts

B is approximate. We used the infinite-solenoid formula. At the center of the solenoid with leng to-diameter ratio $30/6 = 5$, the field is smaller by the factor $\cos(\tan^{-1} 0.2) = 0.98$, or $B_0 = 510$ gauss. (see Eq. 44)

.8



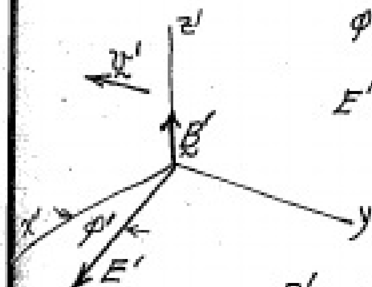
In frame F, $E_x = 100\cos 30° = 86.6$; $E_y = 100 \sin 30° = 50$; $E_z = 0$; $\underset{\sim}{B} = 0$

Eq. 58 is the transformation for motion of F' in the x-direction. For F' moving in the y-direction, with $\underset{\sim}{B} = 0$ in F, we have: $E_y' = E_y \quad E_x' = \gamma E_x \quad E_z' = \gamma E_z = 0$

ere $\gamma = \frac{1}{\sqrt{1 - 0.6^2}} = 1.25$ $\quad E_y' = 50 \quad E_x' = 86.6 \times 1.25 = 108.3$

$$\varphi' = \tan^{-1}\frac{50}{108.3} = 24.8°$$

$$E' = \sqrt{108.3^2 + 50^2} = 119.3 \ \frac{\text{statvolts}}{\text{cm}}$$



To find $\underset{\sim}{B}'$ we can use Eq. 61 with $\underset{\sim}{v}' = -.6c\,\hat{\underset{\sim}{y}}'$

$\underset{\sim}{B}' = \frac{\underset{\sim}{v}'}{c} \times \underset{\sim}{E}'$ $\quad \underset{\sim}{B}'$ is in the direction

$\hat{z}'$, with magnitude $0.6\,E_x' = 0.6 \times 108.3 = 65$ gauss

.9 $\quad \frac{v}{c}B = \frac{100}{3\times 10^{10}} \times 0.35 = 1.17 \times 10^{-9}$ statvolts/cm

r 3.5×10^{-7} volts/cm.

$= E\sigma = 3.5 \times 10^{-7} \times 0.04 = 1.4 \times 10^{-8}$ amp/cm^2

$= 1.4 \times 10^{-4}$ amp/m^2

a bottle of sea water were carried at this eed, a current would flow only long enough to parate enough charge to establish an electric field al and opposite to $\frac{\underset{\sim}{v}}{c} \times \underset{\sim}{B}$. To find how long this es consider the charge that would pile up on one of surface in τ seconds, with $J = 1.4 \times 10^{-8}$ amp/cm^2 $.4 \times 10^{-8} \times 3 \times 10^9$ esu/sec. $= 42$ esu/sec/cm^2. The resulting ld will be $E = 4\pi \times 42\,\tau$, which will equal 10^{-9} tvolts/cm when $\tau = \frac{10^{-9}}{4\pi \times 42} = 2 \times 10^{-12}$ sec.

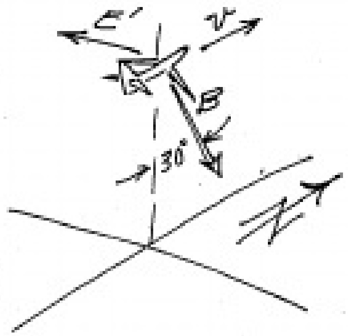
6.10 $\mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B} \qquad \mathbf{B}' = \mathbf{B} - \frac{\mathbf{v}}{c}\times\mathbf{E}$

For $\mathbf{v} = v\hat{\mathbf{x}}$ this becomes:

$$E'_x = E_x \qquad E'_y = E_y - \frac{v}{c}B_z \qquad E'_z = E_z + \frac{v}{c}B_y$$

$$B'_x = B_x \qquad B'_y = B_y + \frac{v}{c}E_z \qquad B'_z = B_z - \frac{v}{c}E_y$$

Except for the absence of the factor γ these agree with Eq. 58.



Assume $v = 500$ mph $= 800$ km/hr

$$v = \frac{800\times 10^5}{3600} = 2.2\times 10^4 \text{ cm/sec}$$

$$\frac{v}{c} = 7\times 10^{-7}$$

$$E' = \frac{v}{c}B\cos 30^\circ = 2.4\times 10^{-7}\ \frac{\text{statvolt}}{\text{cm}}$$

$\mathbf{E}'$ points west.

6.11 $\mathbf{E}'\cdot\mathbf{B}' = E'_x B'_x + E'_y B'_y + E'_z B'_z$

$$= E_x B_x + \gamma^2(E_y B_y - \beta B_y B_z + \beta E_y E_z - \beta^2 E_z B_z)$$
$$+ \gamma^2(E_z B_z + \beta B_y B_z - \beta E_y E_z - \beta^2 E_y B_y)$$
$$= E_x B_x + \gamma^2(1-\beta^2)(E_y B_y + E_z B_z) = E_x B_x + E_y B_y + E_z B_z = \mathbf{E}\cdot\mathbf{B}$$

$$E'^2 - B'^2 = E'^2_x + E'^2_y + E'^2_z - B'^2_x - B'^2_y - B'^2_z$$
$$= E_x^2 + \gamma^2(E_y^2 + \beta^2 B_z^2 - 2\beta E_y B_z) + \gamma^2(E_z^2 + \beta^2 B_y^2 + 2\beta E_z B_y)$$
$$- B_x^2 - \gamma^2(B_y^2 + \beta^2 E_z^2 + 2\beta E_z B_y) - \gamma^2(B_z^2 + \beta^2 E_y^2 - 2\beta E_y B_z)$$
$$= E_x^2 + \gamma^2(1-\beta^2)(E_y^2 + E_z^2) - B_x^2 - \gamma^2(1-\beta^2)(B_y^2 + B_z^2) = E^2 - B^2$$

6.12 $B_x = 0 \quad B_y = 0 \quad B_z = B_0$ with $\mathbf{B} = \nabla\times\mathbf{A}$

$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = 0 \qquad \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = 0 \qquad \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = B_0$$

One obvious choice is: $A_x = -\frac{B_0 y}{2}$, $A_y = \frac{B_0 x}{2}$ $A_z = 0$.

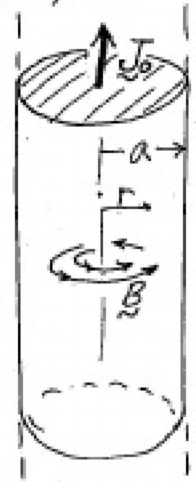
Equally obvious is: $A_y = B_0 x \quad A_x = A_z = 0$. To make others add any vector function with zero curl.

6.13 Stokes' Theorem says:

$$\int_{\text{curve}} \mathbf{A}\cdot d\mathbf{s} = \int_{\text{surface}} (\text{curl}\,\mathbf{A})\cdot d\mathbf{a}, \quad \text{for any } \mathbf{A}.$$

If curl $\mathbf{A} = \mathbf{B}$, then $\int_{\text{curve}} \mathbf{A}\cdot d\mathbf{s} = \int_{\text{surface}} \mathbf{B}\cdot d\mathbf{a} = \Phi$

The line integral of the vector potential $\mathbf{A}$ around any closed path is equal to the flux enclosed by that path.

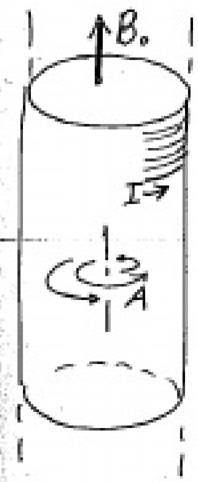


Solid conducting cylinder. $\mathbf{J}$ is constant over the cross-section and zero outside the cylinder.

curl $\mathbf{B} = \frac{4\pi\mathbf{J}}{c}$, and we know that

$$B = \frac{2}{rc}\cdot\pi r^2 J_0 = \frac{2\pi r J_0}{c} \quad \text{for } 0 \le r \le a$$

$$B = \frac{2\pi a^2 J_0}{cr} \quad \text{for } r \ge a$$



Cylindrical solenoid. $\mathbf{B}$ is constant over the cross-section and zero outside the cylinder.

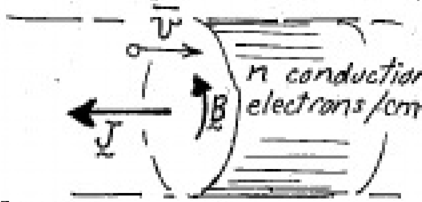
curl $\mathbf{A} = \mathbf{B}$ By analogy with the system above, replacing B by A, and $\frac{4\pi J}{c}$ by B we have:

$$A = \frac{B_0 r}{2} \quad \text{for } 0 \le r \le a$$

$$A = \frac{a^2 B_0}{2r} \quad \text{for } r \ge a$$

The lines of $\mathbf{A}$ are circles around the axis. Of course, this is not the only $\mathbf{A}$ which would satisfy curl $\mathbf{A} = \mathbf{B}$.

6.14 Assume first that the current density J is uniform inside the conductor. Then $B = \frac{2\pi r J}{c}$. An electron with average velocity $\bar{v}$ experiences a force of magnitude $e\frac{\bar{v}}{c}B$ directed radially inward.



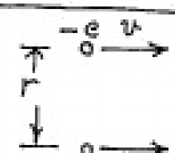
$F = \frac{e\bar{v}}{c}B = \frac{e\bar{v}}{c}\cdot\frac{2\pi r J}{c}$. But $J = ne\bar{v}$, so that

$F = 2\pi n e^2 \frac{\bar{v}^2}{c^2} r$. Let's see what redistribution of charge would be required to produce an electric field that would just balance this inward force. Notice that F is proportional to r. We know that a cylindrical charge distribution has, in its interior, a radial field proportional to r (see Fig. 2.20, p. 59). $E = 2\pi\rho r$. To make $eE = F$ requires

$e\cdot 2\pi\rho r = 2\pi n e^2 \frac{\bar{v}^2}{c^2} r$, or $\rho = ne\left(\frac{\bar{v}}{c}\right)^2$. Now ne is the charge density we would find if all the conduction electrons were present alone, uncompensated by positive charge. The average velocity $\bar{v}$ is typically very small compared to c. See Prob. 4.5 for an example. Or consider that with 10^{22} conduction electrons per cm^3, a drift velocity of 1 cm/sec corresponds to a current density of $10^{22}\times 1.6\times 10^{-19} = 1600$ amps/cm^2. In that case, $(\bar{v}/c)^2 \approx 10^{-21}$. Conclusion: equilibrium requires a very slight redistribution of the conduction electrons, amounting to a uniform increase, by the fraction $(\bar{v}/c)^2$, in the concentration of conduction electrons over the value that exactly neutralizes the positive charge.

If the positive charges were mobile too, the entire distribution could collapse toward the axis without any charge separation. Such an effect can be observed in the conduction of very intense currents in a column of ionized gas. It is called the "pinch effect."

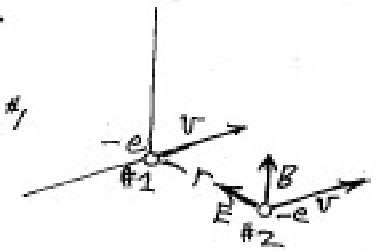
6.15



(a) In a frame of reference moving with the electrons we have two charges at rest, r cm apart. The force is repulsive and its magnitude is simply $\frac{e^2}{r^2}$. Transforming the force to the lab frame, using $\frac{dp_\perp}{dt} = \frac{1}{\gamma}\frac{dp_\perp'}{dt'}$, as given on page 170, we have

$$F = \frac{1}{\gamma}\frac{e^2}{r^2}$$

(b) In the lab frame, consider the fields at the location of electron #2 arising from electron #1



$E = \frac{\gamma e}{r^2}$ (from Eq. 5.12, p. 160, with $\theta' = \frac{\pi}{2}$)

$B = \beta E = \beta\gamma\frac{e}{r^2}$ (applying Eq. 61, p. 214)

The resulting force on electron #2 is made up of:

1) an outward force $eE = \gamma\frac{e^2}{r^2}$

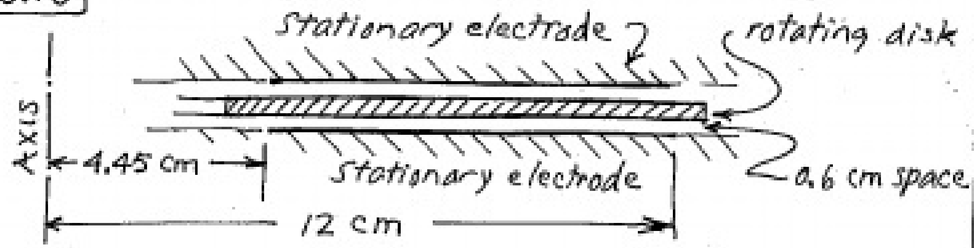
2) an inward force $\frac{ev}{c}B = \frac{e^2\beta^2\gamma}{r^2}$

The net force is $F = \gamma\frac{e^2}{r^2} - \gamma\beta^2\frac{e^2}{r^2} = \gamma(1-\beta^2)\frac{e^2}{r^2} = \frac{1}{\gamma}\frac{e^2}{r^2}$

which agrees with what we found by method (a).

(c) As $v \to c$, $\gamma \to \infty$ and the force, as observed in the lab frame, approaches zero.

6.16

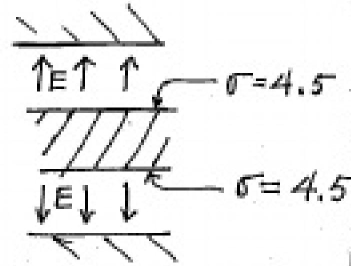


With the disk at 10 kv, the electric field strength in the space above and in the space below the disk is:

$$E = \frac{10^4}{300 \times 0.6} = 55 \text{ statvolts/cm}.$$

Density of surface charge on each surface:

$$\sigma = \frac{E}{4\pi} = 4.5 \text{ esu/cm}^2$$

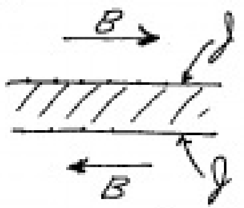


At the mean radius $\bar{r} = \frac{12+4.4}{2}$ $= 8.2$ cm, the velocity is

$$2\pi \times 8.2 \times 61 = 3150 \text{ cm/sec}.$$

The equivalent surface current density is

$$\mathcal{J} = \sigma v = 4.5 \times 3150 = 1.4 \times 10^4 \text{ esu/sec/cm}$$



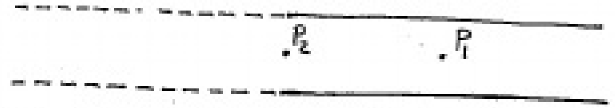
The two current sheets each of surface density $\mathcal{J}$, produce a field $B = \frac{4\pi}{c}\mathcal{J}$ both above and below.

$$B = \frac{4\pi}{c}\mathcal{J} = \frac{4\pi \times 1.4 \times 10^4}{3 \times 10^{10}} = 5.9 \times 10^{-6} \text{ gauss}$$

We have used the velocity at the mean radius to estimate the field strength B immediately above the disk in that region. For a more accurate calculation of the field strength to be expected at the location of the magnetometer needle, one could divide the disk into equivalent circular current rings and integrate over the whole distribution. That is what Rowland did.

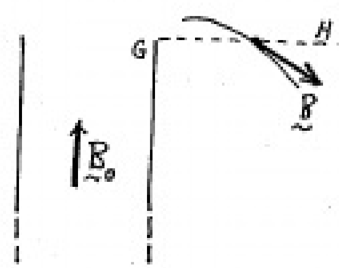
6.17

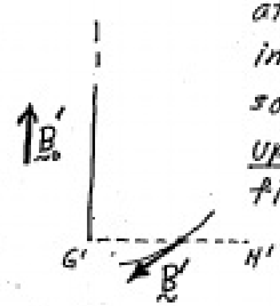
(a)



Imagine adding a similar solenoid on the left, as shown. This exactly doubles the field strength B at P_2. But now the field strengths at P_2 and P_1 are approximately equal, for both points lie well inside a fairly long solenoid, the field at P_2 being slightly stronger. Therefore the original field at P_2 must have been slightly more than half the field at P_1.

(b) Suppose the field does have a vertical component somewhere along the line GH, that is, in the plane of the end of the coil, as shown at the right. Now imagine a similar solenoid, extending <u>upward</u>, as shown at the left. The field $\underset{\sim}{B}'$ at the corresponding point must have a <u>down</u>ward component, if $\underset{\sim}{B}$ does and if $\underset{\sim}{B}_0'$ and $\underset{\sim}{B}_0$ are in the same direction. Now join the two semi-infinite solenoids end to end, and superpose their fields. The resulting infinitely long solenoid must have zero external field. But the addition of $\underset{\sim}{B}$ and $\underset{\sim}{B}'$ cannot give zero. Hence $\underset{\sim}{B}$ cannot have a vertical component. Therefore $\underset{\sim}{B}$ is horizontal at all points on the line GH, and GH is a field line.



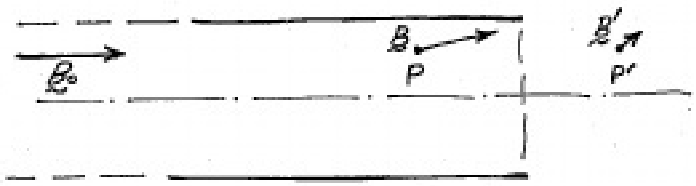


The argument used in (a), applied to the semi-infinite solenoid, shows that the <u>axial</u> component of field, at <u>any</u> point on the end face is $\frac{1}{2}B_0$. Adding another semi-infinite solenoid just doubles the axial field component and produces the purely uniform field B_0. In calculating the flux through

the end face, only the axial field component is involved. Therefore the flux must be just half the interior flux.

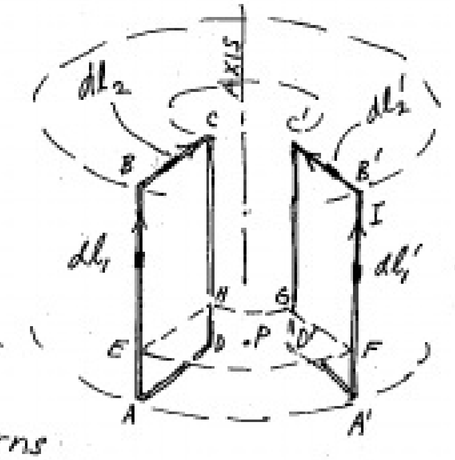
(d) For the same reason, the flux tube on the surface of which the field line CDE lies must flare out as it approaches the end face so that its cross-section there becomes a circle of twice the area containing the same amount of flux. $\pi r^2 = 2\pi r_0^2$, or $r = \sqrt{2}\, r_0$

The arguments used in (b) and (c) lead to a more general statement about the field of the semi-infinite solenoid:

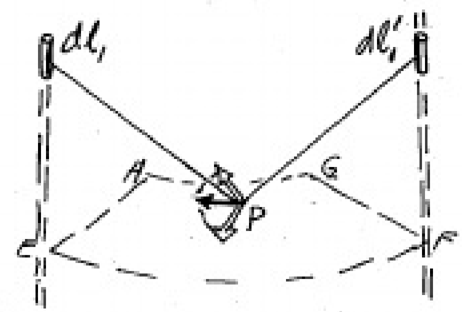


At two corresponding points P and P', symmetrically located with respect to the end plane and equidistant from the axis, the fields $\mathbf{B}$ and $\mathbf{B}'$ are related as follows: The radial components of $\mathbf{B}$ and $\mathbf{B}'$ are equal. The sum of the axial components of $\mathbf{B}$ and $\mathbf{B}'$ is equal to B_0, if P lies inside the coil, or to zero, if P lies outside the coil. The conclusions of (b) and (c) follow in the special case in which P and P' coincide.

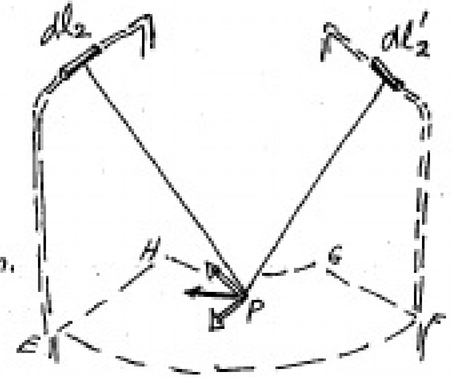
6.19 Consider any point such as P. We want to show that the field $\mathbf{B}$ at P is "circumferential", that is, it lies in the plane EFGH, a plane containing P and perpendicular to the axis, and it is perpendicular to the radius from P to the axis. Consider a pair of turns ABCD and A'B'C'D' symmetrically located with respect to P.



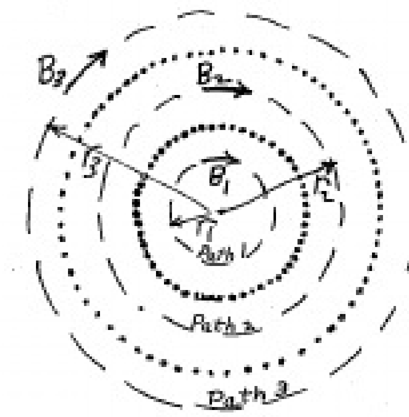
(i) The contributions to the field at P from equal, symmetrically located wire-elements dl_1 and dl_1' combine to form a resultant in the circumferential direction. (Note that each vector lies in the plane EFGH.)



(ii) The contributions to the field at P from equal, symmetrically located wire-elements dl_2 and dl_2' also combine to form a resultant in the circumferential direction. Here the individual vectors are out of the plane, and the geometry is a little harder to visualize.



Since the two rectangular turns can be entirely decomposed into such elements, and the entire coil can be decomposed into such pairs of turns, it follows that the total field at P must be circumferential.

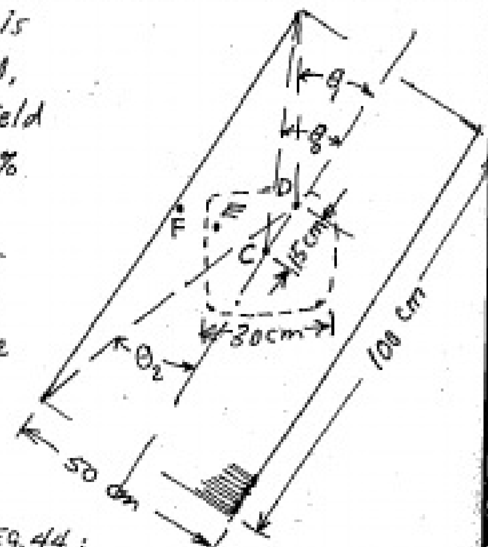


Path 1 and path 3 enclose zero current. Since B, if it exists, is circumferential, the line integral around such a path is $2\pi r B$. Then we must have:

$$B_1 = 0 = B_3 \quad \text{and}$$

$$2\pi r_2 B_2 = \frac{4\pi NI}{c} \quad \text{or} \quad B_2 = \frac{2NI}{r_2 c}$$

6.20 Since 10 milligauss is about 2% of the earth's field, we need a compensating field that is uniform to about 2% over the region of interest. Let's try a solenoid 1 meter long and 50 cm in diameter. To see whether it meets the requirement, compare the field at the center C with the field on the axis 15 cm from the center, at D. From Eq. 44:



$$\frac{\text{field at } C}{\text{field at } D} = \frac{2\cos\theta_0}{\cos\theta_1 + \cos\theta_2}$$

$\theta_0 = \tan^{-1}\frac{25}{50}$ $\qquad \cos\theta_0 = 0.8944$

$\theta_1 = \tan^{-1}\frac{25}{35}$ $\qquad \cos\theta_1 = 0.8138$

$\theta_2 = \tan^{-1}\frac{25}{65}$ $\qquad \cos\theta_2 = 0.9334$

$$\frac{2\cos\theta_0}{\cos\theta_1 + \cos\theta_2} = \frac{1.7888}{1.7472} = 1.024$$

This is a little too large for comfort, especially as we have no easy way to estimate the deviation at off-axis points, such as E. Let's lengthen the solenoid to 120 cm. Then $\theta_0 = \tan^{-1}\frac{25}{60}$, $\theta_1 = \tan^{-1}\frac{25}{45}$ and $\theta_2 = \tan^{-1}\frac{25}{75}$. This gives:

$$\frac{2\cos\theta_0}{\cos\theta_1 + \cos\theta_2} = \frac{1.8462}{1.8229} = 1.012$$

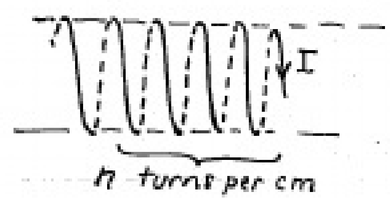
We may expect the departure from uniformity in the radial direction to be of the same magnitude, roughly, as the variation in the axial direction — although it has the opposite sign, of course, the field strength at E being greater than that at C. [Note: an exact calculation of the field strength at F, which involves an elliptic integral, shows that it is 1.4% greater than the field strength at C, in the 120 cm long solenoid.]

The ampere-turns, NI, required to make the field of the solenoid at C equal to the earth's field, 0.55 gauss, are computed as follows:

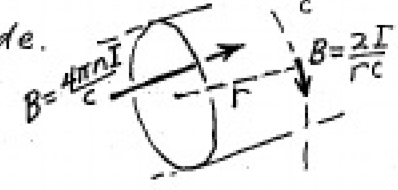
$$\left(\frac{4\pi}{10}\right)\frac{NI}{120}\cos\theta_0 = 0.55$$

$$NI = \frac{0.55 \times 120 \times 10}{4\pi \times 0.923} = 57.0 \text{ ampere turns}$$

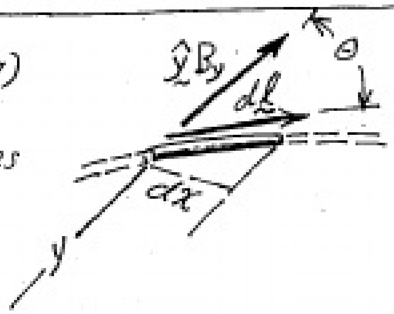
6.21 The longitudinal current flow is the amount of charge passing, per second, through a plane perpendicular to the axis of the coil. This is simply I, for any such plane is pierced just once by the helical wire. Treating this as a cylindrical tube of current, its field will have the magnitude $\frac{2I}{rc}$ outside the tube, and zero inside. The circumferential current produces a uniform axial field of magnitude $\frac{4\pi n I}{c}$ inside the coil, and zero outside.



Close to the individual turns, of course, the field is complicated.



6.22 $d\mathbf{F} = \dfrac{I\, d\mathbf{l} \times \mathbf{B}}{c}$ (Eq. 7)



The z-component of $\mathbf{B}$ produces a force in the plane of the coil, which contributes nothing to the torque and can therefore be ignored in this calculation.

$d\mathbf{l} \times \hat{\mathbf{y}} B_y = \hat{\mathbf{z}}\, dl\, B_y \sin\theta$ But $dl \sin\theta = dx$

Hence $dF_z = \dfrac{I}{c} B_y\, dx$ The torque about the x-axis is $y \cdot dF_z = \dfrac{IB_y}{c}\, y\, dx = dN$. We have to integrate this around the loop to find the total torque.

$$\int_{loop} y\, dx = \text{area of loop} \equiv a$$

(See Fig. 10.4 c, page 362) Hence $N = \dfrac{B_y I a}{c}$. The torque vector $\mathbf{N}$ is in the $\hat{x}$ direction, and if we define $\mathbf{m} \equiv \dfrac{I\mathbf{a}}{c}$ as in the Figure, our result can be written in the more general form: $\mathbf{N} = \mathbf{m} \times \mathbf{B}$.

If $\mathbf{B}$ is the same at all points on the loop, the net force on the loop is zero, for

$$\int d\mathbf{F} = \int \frac{I}{c}\, d\mathbf{l} \times \mathbf{B} = -\frac{I}{c}\mathbf{B} \times \int d\mathbf{l} \quad \text{and}$$

$\int d\mathbf{l}$ over the whole loop is zero, being nothing but the vector displacement between the starting point and ending point of the path.

6.23 For a 1 Bev hydrogen ion, $\gamma \approx 2$, since the total energy, kinetic energy + rest energy, is approximately twice the rest energy.

If $\gamma = 2$, $\beta = \sqrt{1 - \frac{1}{\gamma^2}} = \frac{\sqrt{3}}{2}$

Let E' be the electric field in a coordinate frame moving, at a given instant, with the ion.

$E' = \gamma\beta B = \sqrt{3}\, B$ so that if we require that $E' < 1.5 \times 10^4$ statvolts/cm we must have

$$B < \frac{1.5 \times 10^4}{\sqrt{3}} \quad \text{or} \quad B < 8650 \text{ gauss},$$

6.24 $\mathbf{E}' = \mathbf{E} + \dfrac{\mathbf{v}}{c} \times \mathbf{B}$. Let's take $v = 10^4 \frac{cm}{sec}$, a reasonable speed for an airplane, and $B = 0.5$ gauss. Then the second term on the right is, at most,

$$\frac{10^4 \times 0.5}{3 \times 10^{10}} \approx 2 \times 10^{-7} \text{ statvolts/cm.}$$

Therefore, unless we are provided with advance knowledge of $\mathbf{E}$, <u>to this precision</u>, we cannot infer it from a measurement of $\mathbf{E}'$. Because the variations in the atmospheric electric field from place to place and from day to day are thousands of times greater than this, the situation is hopeless. The pigeon, flying more slowly, is even worse off!

As for measuring $\mathbf{B}'$ and inferring $\mathbf{B}$, the error introduced by unknown electric fields will be of order of magnitude $\frac{v}{c}E$. With $E = 10^{-2}$ statvolts/cm and $v = 10^4$ cm/sec, this is only $\frac{10^4 \times 10^{-2}}{3 \times 10^{10}}$ or 3×10^{-9} gauss, which is entirely negligible.

6.25 Let v be the instantaneous velocity of the bar.

$$|\mathcal{E}| = \left|\frac{1}{c}\frac{d\Phi}{dt}\right| = \frac{1}{c}Bbv \qquad I = \frac{|\mathcal{E}|}{R} = \frac{Bbv}{Rc}$$

The force on the bar: $F = \frac{IbB}{c} = \frac{b^2B^2v}{Rc^2}$

$F = -m\frac{dv}{dt}$ (minus because the force opposes the motion)

$$\frac{dv}{v} = -\frac{B^2b^2}{Rmc^2}dt \qquad \text{Integrating both sides:}$$

$$\ln v = -\frac{B^2b^2}{Rmc^2}t + \text{constant.} \qquad \text{If } v = v_0 \text{ at}$$

$t=0$, we have: $v = v_0 e^{-t/T}$, where $T = \frac{Rmc^2}{B^2b^2}$

The velocity decreases exponentially — in that sense, the rod never stops moving. But the distance it travels is finite:

$$x = \int_0^\infty v\,dt = \int_0^\infty v_0 e^{-t/T}dt = Tv_0$$

The initial kinetic energy of the rod, $\frac{1}{2}mv_0^2$, is transferred to the resistor as heat:

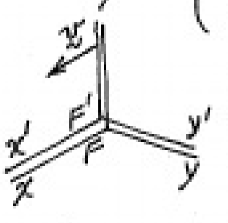
$$\frac{1}{2}mv_0^2 = \int_0^\infty RI^2dt \quad , \quad \text{as can be easily}$$

verified: $I = I_0e^{-t/T} = \frac{Bbv_0}{Rc}e^{-t/T}$

$$\int_0^\infty RI^2dt = \frac{B^2b^2v_0^2}{Rc^2}\int_0^\infty e^{-2t/T}dt = \frac{B^2b^2v_0^2}{Rc^2}\cdot\left(\frac{T}{2}\right) = \frac{mv_0^2}{2}$$

6.26

Eq. 58 $\begin{cases} E_x' = E_x \;;\; E_y' = \gamma(E_y - \beta B_z) \;;\; E_z' = \gamma(E_z + \beta B_y) \\ B_x' = B_x \;;\; B_y' = \gamma(B_y + \beta E_z) \;;\; B_z' = \gamma(B_z - \beta E_y) \end{cases}$



The unique direction is that of $\underset{\sim}{v}$, which is the velocity of the frame F' as observed in the frame F. Given any electric field $\underset{\sim}{E}$ and magnetic field $\underset{\sim}{B}$ in the F frame, we may decompose these vectors into "longitudinal" and "transverse" components, with respect to the direction of $\underset{\sim}{v}$. That is, we define $E_\parallel$ and $E_\perp$ as follows:

$$\underset{\sim}{E}_\parallel = \hat{\underset{\sim}{v}}(\hat{\underset{\sim}{v}}\cdot\underset{\sim}{E}) \qquad \underset{\sim}{E}_\perp = \underset{\sim}{E} - \underset{\sim}{E}_\parallel$$

and similarly decompose $\underset{\sim}{B}$ into $\underset{\sim}{B}_\parallel$ and $\underset{\sim}{B}_\perp$. Then the transformation can be written:

$$\underset{\sim}{E}_\parallel' = \underset{\sim}{E}_\parallel \qquad \underset{\sim}{E}_\perp' = \gamma\left(\underset{\sim}{E}_\perp + \frac{\underset{\sim}{v}}{c}\times\underset{\sim}{B}\right)$$

$$\underset{\sim}{B}_\parallel' = \underset{\sim}{B}_\parallel \qquad \underset{\sim}{B}_\perp' = \gamma\left(\underset{\sim}{B}_\perp - \frac{\underset{\sim}{v}}{c}\times\underset{\sim}{E}\right)$$

(Note that we could just as well write $\underset{\sim}{B}_\perp$ in place of $\underset{\sim}{B}$, or $\underset{\sim}{E}_\perp$ in place of $\underset{\sim}{E}$ in the final terms, since $\underset{\sim}{v}\times\underset{\sim}{B}_\parallel = 0$ and $\underset{\sim}{v}\times\underset{\sim}{E}_\parallel = 0$.)

To go back to the particular case with $\underset{\sim}{v}$ in the $\hat{y}$ direction, we put $\underset{\sim}{v} = \beta c\hat{\underset{\sim}{y}}$, $E_\parallel = E_y$, etc.

$$E_y' = E_y \qquad \underset{\sim}{E}_\perp' = \hat{\underset{\sim}{x}}E_x' + \hat{\underset{\sim}{z}}E_z' = \gamma(\underset{\sim}{E}_\perp + \beta\hat{\underset{\sim}{y}}\times\underset{\sim}{B})$$

$$= \gamma\underset{\sim}{E}_\perp + \beta\gamma(\hat{\underset{\sim}{x}}B_z - \hat{\underset{\sim}{z}}B_x)$$

Thus:

$$E_y' = E_y \qquad E_x' = \gamma(E_x + \beta B_z) \qquad E_z' = \gamma(E_z - \beta B_x)$$

and

$$B_y' = B_y \qquad B_x' = \gamma(B_x - \beta E_z) \qquad B_z' = \gamma(B_z + \beta E_x)$$

6.27 A frame moving in the x direction with speed v such that $\beta B_z = E_y$ gives (see Eq. 58)

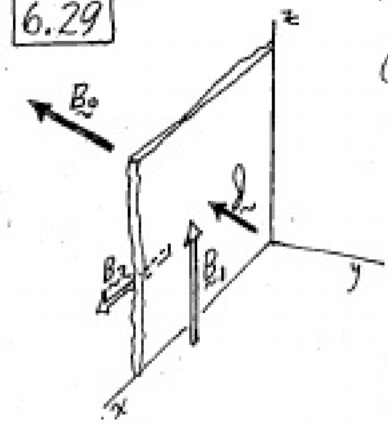
$$E_y' = \gamma(E_y - \beta B_z) = 0 \;,\; E_x' = E_z' = 0$$

If $E_y > B_z$ there is no frame in which $\underset{\sim}{E}' = 0$, but there is a frame in which $\underset{\sim}{B}' = 0$, namely one moving with speed $v = \frac{B_z}{E_y}c$ in the x-direction.

Notice that in the special case $E_y = B_z$, in a frame moving with speed c in the x direction there is no field at all: $\underset{\sim}{E}' = \underset{\sim}{B}' = 0$. This relation is characteristic of the fields in a light wave, as we shall see in Chapter 7, and the transformation tells us, "If you could catch up to a light wave, you wouldn't find anything."

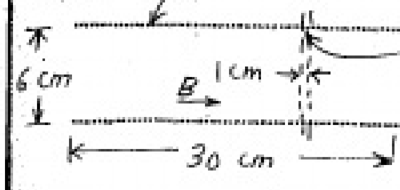
6.28 The analysis in Sec. 5.9 showed that a test charge moving parallel to a wire carrying current experiences a force which, as observed in the rest frame of the test charge, is due to an electric field. To understand why the introduction of a conductor, such as a metal plate, between the wire and the test charge, has no effect, let us view the situation from the rest frame of the test charge. In that frame, the conducting plate, which is stationary in the lab frame, is moving. It is moving through a magnetic field and an electric field and these are related precisely so as to make the total force on any charge in the plate zero. Hence there is no redistribution whatever of the electrons in the plate. On the other hand, if we caused the plate to move along with the same velocity as the test charge, it would make a difference. An observer in the test-charge frame would say that we have introduced a stationary plate into an electrostatic field, with a consequent redistribution of charge on the plate and a resulting alteration of the total electric field. An observer in the lab frame, where there was no electric field before, would say that the electrons in the plate have redistributed themselves under the influence of the $q\frac{v}{c}\times B$ force, and the new charge distribution itself produces an electric field.

6.29

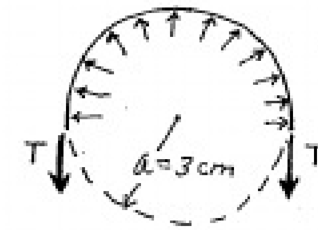


The statement on pp. 209–210 that $(B_1^2 - B_2^2)/8\pi$ is the force per cm² would imply that the force is zero in this case. Indeed that is true, for what we have here is the superposition of an external field (i.e., a field from some other source) B_0 in the direction $\hat{y}+\hat{z}$, and the field of current sheet $\underline{J}$ flowing in the same direction.

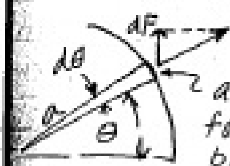
6.30 #14 wire, 0.163 cm diameter, 5 turns per cm.



Consider a band 1 cm wide, and isolate a semi-circular half of it



The magnetic pressure, $\frac{B^2}{8\pi}$ dynes/cm², acting radially outward on the conductors, causes an upward force on the semicircle which is balanced by the downward force 2T. This upward resultant F is:



area = a dθ for 1 cm wide band

$$F = \int_0^{\pi} \frac{B^2}{8\pi} a\, d\theta \sin\theta = \frac{aB^2}{4\pi}$$

For equilibrium, then, $2T = \frac{aB^2}{4\pi}$ or $T = \frac{aB^2}{8\pi}$

The force T is distributed over 5 wires, hence over cross-sectional area of $5 \times \frac{\pi}{4} \times 0.163^2 = 0.104$ cm²

If the limiting stress is 2×10^9 dynes/cm², we have:

$$2\times10^9 = \frac{T}{0.104} = \frac{3B_{max}^2}{0.104 \times 8\pi}$$ which gives $B_{max} = 42$ kilogauss

6.31 $E = \frac{1}{nqc} JB$ (Eq. 15, p. 220)

This is in Gaussian CGS units. Converting the data given: 25×10^{-6} volts/cm $= \frac{25\times10^{-6}}{300} = 8.33\times10^{-8}$ statvolts/cm

$J =$ 1000 amp/cm² $= 3\times10^9 \times 1000 = 3\times10^{12}$ esu/cm²/sec

$$nq = \frac{JB}{Ec} = \frac{3\times10^{12}\times10^4}{8.33\times10^{-8}\times3\times10^{10}} = 1.2\times10^{13} \text{ esu/cm}^3$$

If $q = e$, $n = \frac{1.2\times10^{13}}{4.8\times10^{-10}} = 2.5\times10^{22}$ /cm³

Number of atoms in 1 cm³ in Na (At. wt = 23):

$$n = \frac{.97\times6\times10^{23}}{23} = 2.54\times10^{22}/\text{cm}^3$$

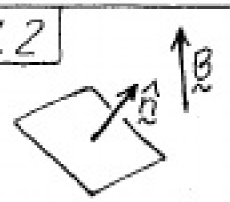
CHAPTER 7

7.1 Turns × area = $4000 \times \pi \times 12^2 = 1.81 \times 10^6$ turns-cm²

$\omega = 2\pi \times 30 = 188$ radians/sec $B = 0.5$ gauss

$$\mathcal{E}\text{ (volts)} = 10^{-8}\frac{d\Phi}{dt} = 10^{-8} \times 0.5 \times 188 \times 1.81 \times 10^6 = 1.70 \text{ volts}$$

7.2



Let $\hat{n}$ be the unit vector normal to the plane of the loop. The flux through the loop will remain constant and the induced electromotive force will be zero for any motions which preserve the angle between $\hat{n}$ and the fixed direction $\mathbf{B}$. These include: (1) translation in any direction; (2) rotation about an axis parallel to $\hat{n}$; (3) rotation about an axis parallel to $\mathbf{B}$.

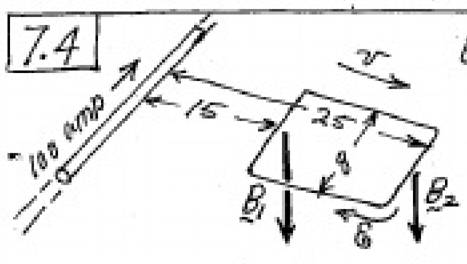
7.3 $\mathcal{E} = -\frac{1}{c}\frac{d\Phi}{dt}$ $I = \frac{\mathcal{E}}{R} = -\frac{1}{cR}\frac{d\Phi}{dt}$

$$Q = \int I\,dt = -\frac{1}{cR}\int d\Phi = -\frac{1}{cR}(\Phi_f - \Phi_i)$$

In the 90° flip, $\Phi_i = NAB$, $\Phi_f = 0$ $Q = \frac{NAB}{cR}$

In the 180° flip, $\Phi_i = NAB$, $\Phi_f = -NAB$ $Q = \frac{2NAB}{cR}$

7.4



$B = \frac{2}{10}\frac{I}{r} = \frac{20}{r}$ gauss

$B_1 = \frac{20}{15} = 1.33$ gauss

$B_2 = \frac{20}{25} = 0.8$ gauss

$v = 500$ cm/sec

$$\mathcal{E}\text{ (volts)} = 10^{-8}\frac{d\Phi}{dt} = 10^{-8} \times 8 \times 500\,(1.33 - 0.8) = 2.13 \times 10^{-5} \text{ volts}$$

The flux is downward and is decreasing. $\mathcal{E}$ will be in the direction to drive a current which would make more flux downward, that is:



Assume the current in the loop, at any instant, is $\mathcal{E}/R$. This current I causes a field B' and a flux Φ' linking the loop. Because $\mathcal{E}$ is changing with time as the loop moves away from the wire, Φ' is changing too, resulting in an extra induced emf $\mathcal{E}'$, which we have ignored. The question is how large must R be so that $\mathcal{E}'$ is indeed negligible compared to $\mathcal{E}$? As a very rough estimate, $B' \approx \frac{I'}{10}\cdot\frac{1}{5}$ (taking 5 cm as a typical dimension of the loop) and $\Phi' \approx B' \times$ loop area $\approx \frac{I'}{50} \times 8 \times 10 \approx 2I'$. (Strictly, we should expect a factor like $\ln\left(\frac{\text{loop diameter}}{\text{wire diameter}}\right)$ to come in when the flux of B' linking the loop is calculated – see the comments in the first paragraph on p. 252 – but unless the wire is extremely thin the logarithm won't be a very large number.) The time characteristic of these changes is of the order of magnitude of $\frac{\text{mean distance to wire}}{\text{velocity of loop}} \approx \frac{20}{500}$ sec $= 0.04$ sec. Thus

$$\mathcal{E}' = 10^{-8}\frac{d\Phi'}{dt} \approx 10^{-8}\cdot\frac{2I'}{.04} = 5 \times 10^{-7}\frac{\mathcal{E}}{R}$$

Thus if $R \gg 10^{-6}$ ohms, we will have $\mathcal{E}' \ll \mathcal{E}$. To state it more generally, the inductive time constant of the loop itself, L/R, should be short compared to the time of change of the externally induced emf.

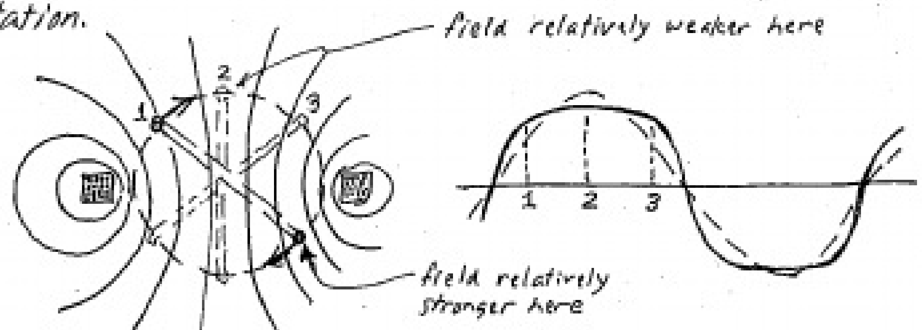
7.5 $\mathcal{E} = \frac{vw}{c}(B_1 - B_2)$ $I = \frac{\mathcal{E}}{R} = \frac{vw}{cR}(B_1 - B_2)$

The force on the loop is: $F = \frac{IB_1w}{c} - \frac{IB_2w}{c} = \frac{Iw}{c}(B_1 - B_2)$

Rate at which work must be done to move the loop is $Fv = \frac{Iwv}{c}(B_1 - B_2) = I^2R$

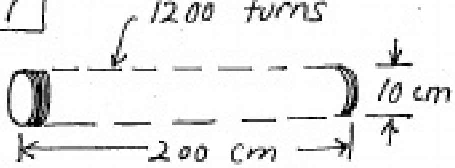
In Fig. 7.14 the energy which is dissipated in the stationary loop has to be supplied by whatever agency is moving the coil. A force is required to move the coil because of the magnetic field arising from the induced current in the loop.

7.6 If the field is uniform, $\mathcal{E}$ will be sinusoidal, regardless of the shape of the loop, at constant rate of rotation.



A loop rotating in this field will generate an emf more like a "square-wave".

7.7



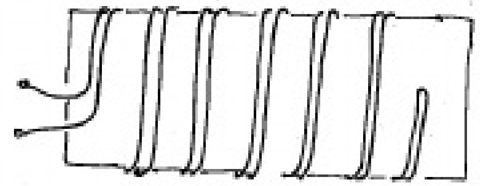
1 ampere in the coil causes a field inside the coil of magnitude $B = \frac{4\pi}{10} \times 1 \times \frac{1200}{200} = 7.55$ gauss

flux × turns = $\Phi N = \pi \times 5^2 \times 7.55 \times 1200 = 7.1 \times 10^5$

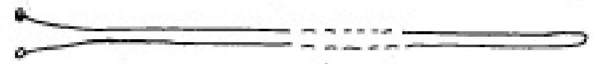
L (henrys) $= 10^{-8} \times 7.1 \times 10^5 = 7.1 \times 10^{-3}$ henrys.

We have neglected the fact that the field inside the solenoid is not constant. It decreases near each end, so that the flux through the last turn is only about half that through a turn in the middle. This means that we have over-estimated the inductance. We might expect the error to be roughly $\frac{\text{diameter}}{\text{length}}$, or ~ 5% in this example. [In fact, the error is only 2% in this case, as one can discover by referring to tables which give exact values for the inductance of cylindrical coils.]

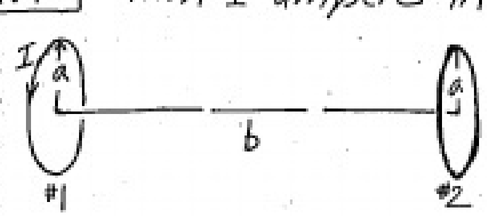
7.8 One way to wind resistance wire into a "non-inductive" coil is indicated below. Of course



the inductance is not zero. The residual inductance is approximately that of the long, narrow "hair-pin" configuration:



7.9 With I amperes in ring #1, the field a distance b down the axis is:



$$B = \frac{2\pi a^2 I}{10(a^2+b^2)^{3/2}}$$

This is an application of Eq. 6.41, p. 202, with a replacing b, b replacing z, and $\frac{1}{10}$ replacing $\frac{1}{c}$ since I is in amperes. For $b \gg a$ this can be approximated:

$$B \approx \frac{2\pi a^2 I}{10\, b^3}$$

and also, for $b \gg a$, we can neglect the variation of B over the interior of ring #2. Then:

$$\Phi_{12} = B \cdot \pi a^2 = \frac{2\pi^2 a^4 I}{10\, b^3}$$

The mutual inductance in henrys is $10^{-8}\frac{\Phi}{I}$, or

$$M = \frac{2\pi^2 \times 10^{-9} a^4}{b^3} \text{ henrys}$$

7.10 We have enough information to calculate the resistance of each coil, if we know the resistivity of copper. From the graph on page 128 we can read off an accurate enough value – say 2×10^{-6} ohm-cm, at room temperature.

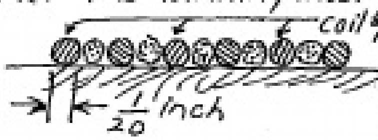
$$\text{wire length} = 203\times12\times2.54 = 6200 \text{ cm}$$

$$\text{cross-section} = \frac{\pi}{4}\left(\frac{2.54}{20}\right)^2 = 0.0126 \text{ cm}^2$$

$$R = \frac{6200\times2\times10^{-6}}{1.26\times10^{-2}} \simeq 1 \text{ ohm}$$

We don't know the coil size or the number of turns. Suppose the coil was cylindrical, with length b and radius a. Then if the length of a wire is ℓ and the number of turns is N, $\ell = 2\pi a N$.

We have another clue: the two coils were wound closely together separated only by twine. The winding must have looked something like this:



If we assume the twine was about as thick as the wire, the turns in one coil were spaced at intervals of $\frac{4}{20}$ inch or about 0.5 cm. Then $N = \frac{b}{0.5} = 2b$

It seems likely that Faraday's "block of wood" would have been roughly "squarish" in proportions. Let's assume $b = 2a$. Then from the three relations,

$\left.\begin{array}{l}\ell = 2\pi a N\\ N = 2b\\ b = 2a\end{array}\right\}$ we find at once: $\ell = 2\pi b^2$
with $\ell = 6200$ this gives $b = 30$ cm,
$2a = 30$ cm and $N = 60$ turns.

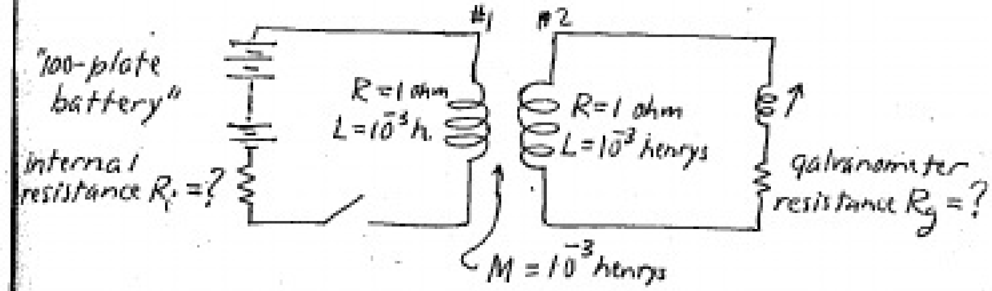
The coil is 1 ft long and 1 ft in diameter – very reasonable.

An approximate formula for the inductance L of a coil of N turns is easily derived. Taking $B \approx \frac{4\pi}{10}\frac{NI\text{(amp)}}{b}$, as for a long solenoid, we have

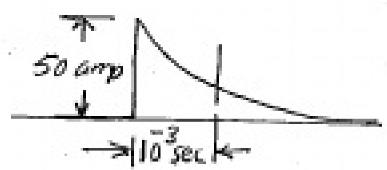
$$\Phi = \pi a^2 B = \frac{4\pi^2 N I a^2}{10\, b} \quad \text{and} \quad L = 10^{-8}\frac{N\Phi}{I} \text{ henrys}$$

$$L \simeq \frac{4\pi^2\times10^{-9}N^2a^2}{b} = \frac{4\pi^2\times10^{-9}\times60^2\times15^2}{30} = 1.1\times10^{-3} \text{ henrys}$$

Because end effects were neglected, this somewhat over-estimates the inductance, so we might as well round it off to $L = 10^{-3}$ henrys. This is also the mutual inductance of the two coils, for they link the same flux, in this approximation. The reconstructed circuit looks like this:



A 100-plate battery would have an emf of the order of 100 volts. Nothing would have been gained by using so large a battery if its internal resistance were much greater than 1 ohm, but it probably wasn't much less. So let's assume $R_i \approx 1$ ohm. Then with the switch closed, the steady current through coil #1 was 50 amperes. When the switch is opened the current must rise instantaneously to 50 amperes in coil #2, for the flux cannot decrease discontinuously. Thereafter the current decays in circuit #2 with the time constant $\frac{L}{R+R_g}$. Assuming $R_g \ll R$ (maybe it wasn't!) we have $\frac{L}{R} = \frac{10^{-3}}{1} = 10^{-3}$ seconds. Then the current pulse looked something like this:



7.11 Referring to (a) in the Figure, if I_2 is increasing we have increasing upward flux through circuit #1. This will induce an $\mathcal{E}_1$ in a direction to drive current to make downward flux, which is opposite to the positive direction assigned to $\mathcal{E}_1$. Hence the equation must be written:

$$\mathcal{E}_1 = -L_1\frac{dI_1}{dt} - M\frac{dI_2}{dt}$$

A similar argument shows that the second equation must be written:

$$\mathcal{E}_2 = -L_2\frac{dI_2}{dt} - M\frac{dI_1}{dt}$$

Had we assigned the opposite positive directions for I_2 and $\mathcal{E}_2$, the signs before M in both equations above would be +.

For the circuit in (b):

$$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 \qquad I = I_1 = I_2$$

$$\mathcal{E} = -L_1\frac{dI}{dt} - M\frac{dI}{dt} - L_2\frac{dI}{dt} - M\frac{dI}{dt} = -(L_1+L_2+2M)\frac{dI}{dt}$$

This is equivalent to a single coil with $L' = L_1+L_2+2M$

For the circuit in (c):

$$\mathcal{E} = \mathcal{E}_1 - \mathcal{E}_2 \qquad I = I_1 = -I_2$$

$$\mathcal{E} = -L_1\frac{dI_1}{dt} - M\frac{dI_2}{dt} + L_2\frac{dI_2}{dt} + M\frac{dI_1}{dt}$$

$$= -L_1\frac{dI}{dt} + M\frac{dI}{dt} - L_2\frac{dI}{dt} + M\frac{dI}{dt} = -(L_1+L_2-2M)\frac{dI}{dt}$$

This is equivalent to a single coil with $L'' = L_1+L_2-2M$. Evidently $L' > L''$. A circuit with $L<0$ would violate Lenz's Law — it would be unstable. This means that L'' must be ≥ 0, which implies $M \leq \frac{L_1+L_2}{2}$ for any pair of circuits. [An even stronger inequality, $M \leq \sqrt{L_1L_2}$, can be derived by considering the coils connected in parallel.]

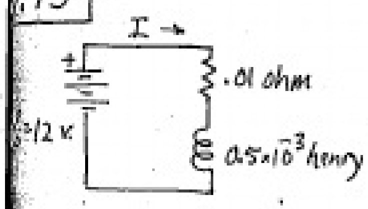
7.12 The magnetic energy density at 10^{-6} gauss is: $\frac{B^2}{8\pi} = \frac{(10^{-6})^2}{8\pi} = 4\times10^{-14}$ ergs/cm³

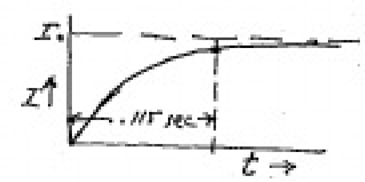
The kinetic energy of a hydrogen atom with a speed of 10^5 cm/sec is:

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}\times1.7\times10^{-24}\times(10^5)^2 = 0.85\times10^{-14}\text{ ergs}$$

With 1 atom/cm³ the kinetic energy density is 0.85×10^{-14} ergs/cm³, somewhat less than, but of the same order of magnitude as the magnetic energy density. (The difference is not significant — the numbers given were too uncertain.)

7.13





$I_0 = \frac{\mathcal{E}}{R}$ = 1200 amp.

$\frac{R}{L} = 20\text{ sec}^{-1}$.

$$I = I_0\left(1-e^{-\frac{R}{L}t}\right) = 1200\left(1-e^{-20t}\right)$$

$$I = 0.9I_0 \quad\text{for}\quad e^{-20t} = 0.1$$

$$20t = \ln_e 10 = 2.30 \qquad t = 0.115\text{ sec}$$

I = 1080 amp at $t = 0.115$

Magnetic field energy $= \frac{1}{2}LI^2 = \frac{1}{2}(0.5\times10^{-3})(1080)^2 = 292$ joules

Energy supplied by battery between $t=0$ and $t=.115$

$$\int_0^{.115}\mathcal{E}I\,dt = \mathcal{E}I_0\int_0^{.115}\left(1-e^{-\frac{R}{L}t}\right)dt = \frac{\mathcal{E}I_0L}{R}\int_0^{2.3}(1-e^{-x})\,dx$$

$$\frac{\mathcal{E}I_0L}{R}\left[x+e^{-x}\right]_0^{2.3} = \frac{\mathcal{E}I_0L}{R}\left[2.3+0.1-1.0\right] = 1.4\,\frac{\mathcal{E}I_0L}{R}$$

$$\frac{1.4\times12\times1200}{20} = 1008\text{ Joules}$$

7.14 If we neglect the edge fields, an approximation which is not very good unless $s \ll b$, the displacement current will be uniformly distributed in the gap, and the total displacement current in the gap will equal the conduction current I in the wire. The fraction of the current enclosed by a circle through P_1 centered on the axis, will be $\pi r^2/\pi b^2$. Hence $2\pi r B = \frac{4\pi}{c}\frac{r^2}{b^2} I$ or

$$B = \frac{2Ir}{cb^2}.$$

7.15 $E_x = E_y = 0 \qquad E_z = \cos(y-ct)$

$B_x = \cos(y-ct) \qquad B_y = B_z = 0$

$$\nabla \times \mathbf{E} = \hat{x}\frac{\partial E_z}{\partial y} = -\hat{x}\sin(y-ct) \qquad \frac{\partial \mathbf{E}}{\partial t} = c\hat{z}\sin(y-ct)$$

$$\nabla \times \mathbf{B} = -\hat{z}\frac{\partial B_x}{\partial y} = \hat{z}\sin(y-ct) \qquad \frac{\partial \mathbf{B}}{\partial t} = c\hat{x}\sin(y-ct)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \quad \text{and} \quad \nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

Also, $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$

7.16 Assume the speed of the tidal current is 1 meter/sec (about 2 knots).

$$\frac{v}{c}B = \frac{10^2}{3\times 10^{10}} \times 0.5 \text{ statvolts/cm} = \frac{300\times 10^2}{3\times 10^{10}} \times 0.5 = 5\times 10^{-7} \text{ volts/cm}$$

960 ft $\approx 3\times 10^4$ cm $\qquad 3\times 10^4 \text{ cm} \times 5\times 10^{-7}$ volts/cm $= 15$ millivolts

7.17 Let s = side of square, in cm $\quad A$ = cross-section in cm^2
D = density of metal, in gm/cm^3 $\quad \rho$ = resistivity, in ohm-cm

resistance of frame: $R = \frac{4s\rho}{A}$ $\qquad$ mass of frame: $m = 4sAD$

If frame velocity is v, $\mathcal{E} = 10^{-8} sBv$, in volts

$$I = \frac{\mathcal{E}}{R} = 10^{-8}\frac{sBv}{R} \text{ amperes} = \frac{10^{-8}}{4}\frac{BvA}{\rho}$$

Force upward $= \frac{I}{10} sB = \frac{10^{-9}}{4}\frac{B^2 A s v}{\rho}$ $\qquad$ velocity remains constant when $F = mg$ or $\frac{10^{-9}}{4}\frac{B^2 A s v_0}{\rho} = 4sADg$

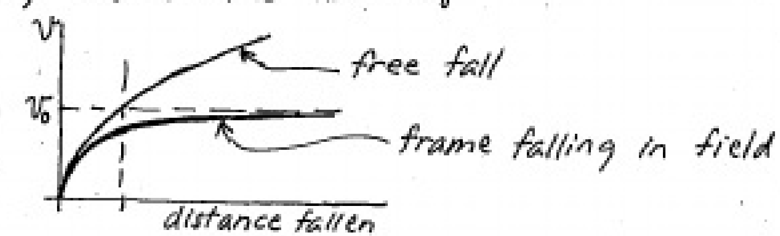
Solving for v_0: $\quad v_0 = 16\times 10^9 D\rho g/B^2$

Putting in the values: $D = 2.7 \quad \rho = 2.8\times 10^{-6} \quad g = 980$ and $B = 1.5\times 10^4$ gives:

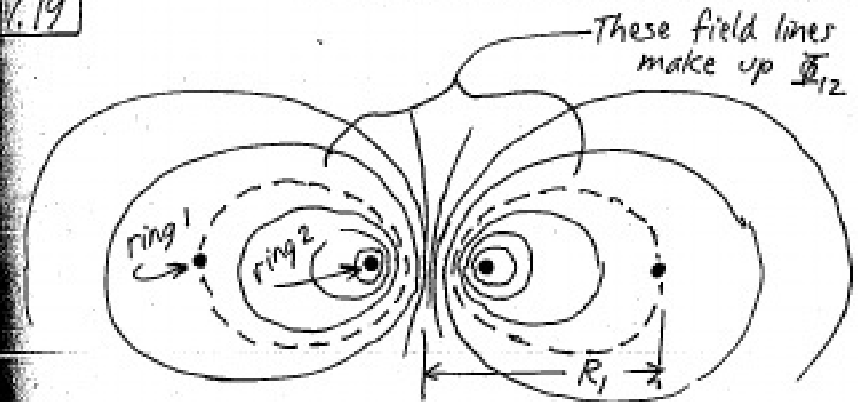
$$v_0 = \frac{16\times 10^9 \times 2.7 \times 2.8\times 10^{-6} \times 980}{2.25\times 10^8} = 0.53 \text{ cm/sec}$$

A freely falling body would acquire speed v after falling a distance $y = \frac{v_0^2}{2g} = \frac{(.53)^2}{2\times 980} \simeq 1.4\times 10^{-4}$ cm.

Here the speed 0.53 cm/sec is approached more slowly, as a limit (in fact exponentially), but this will still be the characteristic distance, so we can expect that after the frame has fallen 10^{-3} cm, say, its speed will be very close to the limit v_0

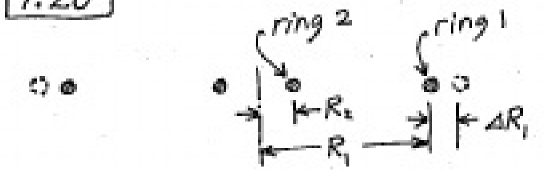


7.19



The point is that the flux which links ring 2 is determined by the lines of force, in the field of ring 1, which return outside ring 2. That is why Φ_{12} decreases as R_1 increases, holding R_2 and I_2 constant.

7.20



With current I in the outer ring the flux through the inner ring, for $R_1 \gg R_2$, is: $\Phi_{21} = \pi R_2^2 \cdot \frac{2\pi I}{cR_1}$

or $\Phi_{21} = \frac{2\pi^2 I}{c}\frac{R_2^2}{R_1}$. Suppose we change R_1 to $R_1 + \Delta R_1$, by expanding the outer ring while holding I constant. The resulting change in Φ_{21} is:

$$\Delta\Phi_{21} = \frac{\partial \Phi_{21}}{\partial R_1}\Delta R_1 = -\frac{2\pi^2 I}{c}\frac{R_2^2}{R_1^2}\Delta R_1$$

Now consider a current I in the inner ring, ring 2. Let B be the field strength at the radius of the outer ring, R_1. If we now expand the outer ring by ΔR_1 the flux Φ_{12} decreases by just the amount of flux between the circle of radius R_1 and the circle of radius $R_1 + \Delta R_1$. (Problem 7.19 explained why it is a decrease.) The change in flux is

$$\Delta\Phi_{21} = -B \cdot 2\pi R_1 \Delta R_1$$

since $2\pi R_1 \Delta R_1$ is the area between the circles.

Our theorem $\Phi_{12} = \Phi_{21}$ guarantees that $\Delta\Phi_{12} = \Delta\Phi_{21}$

Hence:
$$-\frac{2\pi^2 I}{c}\frac{R_2^2}{R_1^2}\Delta R_1 = -B \cdot 2\pi R_1 \Delta R_1$$

Solving for B: $B = \frac{\pi R_2^2 I}{c R_1^3}$ or more generally, $B = \frac{\pi R_2^2 I}{c r^3}$ at any point in the plane of the ring where $r \gg R_2$.

7.21

Assume current I flows in the outer solenoid. The field inside, approximately uniform in the region occupied by the inner solenoid, is $B = \frac{4\pi I}{c}\frac{N_2}{b_2}$.

We have assumed here that $\frac{b_2}{a_2}$ is so large that we can use the formula for an infinite solenoid. We can refine this by using Eq. 6.44, page 203, to calculate the field at the center of a finite solenoid of length b_2, radius a_2. The correction factor is simply $\cos\left(\tan^{-1}\frac{2a_2}{b_2}\right)$ or $b_2/\sqrt{b_2^2 + 4a_2^2}$. This will still not lead us to an exact result, for the inner coil includes a finite volume in which the field strength varies somewhat. But for the proportions shown in the figure, the approximation will be pretty good.

The flux linking the inner coil is

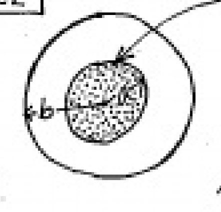
$$\Phi_{12} = \pi a_1^2 B N_1 \quad \text{or} \quad \frac{4\pi^2 I}{c}\frac{N_1 N_2 a_1^2}{b_2}\left(\frac{b_2}{\sqrt{b_2^2 + 4a_2^2}}\right)$$

Since $\mathcal{E}_{12} = -\frac{1}{c}\frac{d\Phi_{12}}{dt} = -M\frac{dI}{dt}$, we have, in CGS units, $M = \frac{4\pi^2 N_1 N_2 a_1^2}{c^2 b_2}\left(\frac{b_2}{\sqrt{b_2^2 + 4a_2^2}}\right)$

To express M in henrys, replace the constant $\frac{1}{c^2}$ by 10^{-9}:

$$M = \frac{4\pi^2 \times 10^{-9} N_1 N_2 a_1^2}{\sqrt{b_2^2 + 4a_2^2}} \text{ henrys}$$

7.22



Current I is uniformly distributed over the cross-section of the inner conductor, and returns in outer conductor whose thickness we neglect.

For $r < a$, $\frac{4\pi I}{c}\frac{r^2}{a^2} = 2\pi r B$, or $B = \frac{2Ir}{ca^2}$

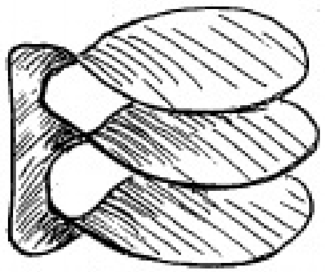
For $a < r < b$, $B = \frac{2I}{cr}$. For $r > b$, $B = 0$

$$\text{Energy stored} = \frac{1}{8\pi}\int B^2 dv = \frac{\ell}{8\pi}\int_0^a \left(\frac{2Ir}{ca^2}\right)^2 2\pi r\,dr + \frac{\ell}{8\pi}\int_a^b \left(\frac{2I}{cr}\right)^2 2\pi r\,dr$$

$$= \frac{\ell I^2}{c^2 a^4}\int_0^a r^3 dr + \frac{\ell I^2}{c^2}\int_a^b \frac{dr}{r} = \frac{\ell I^2}{c^2}\left(\frac{1}{4} + \ln\frac{b}{a}\right) = \frac{1}{2}LI^2$$

Hence $L = \frac{2\ell}{c^2}\left(\frac{1}{4} + \ln\frac{b}{a}\right)$. If the current I is changing very rapidly, the current density will not be uniform in the inner conductor. In the high-frequency limit, the current is all on the surface at $r = a$, and $L \to \frac{2\ell}{c^2}\ln\frac{b}{a}$.

7.23 Surface (a) has two sides. Surface (b) has only one side, it is a "Möbius strip."
The two-sided surface is the one you must use to calculate the flux through the loop. The two-sided surface for a three-turn coil looks like this ⟶ (or could be deformed to look like this). The extension to N turns is obvious.



7.25 Neglecting the inductance and resistance of the two rings and the leads, the charge in the capacitor C_2 at any time t is $Q = \mathcal{E}_0(\cos 2\pi f t)C_2$.
Since $I = \frac{dQ}{dt}$, $I = -2\pi f \mathcal{E}_0 C_2 \sin 2\pi f t$. The two rings are in series and this current flows in each. Assume $h \ll b$ so that we are justified in computing the force between the rings as if they were parallel straight wires, with force per unit length $= \frac{2I^2}{c^2 h}$. The length is $2\pi b$, so the force pulling the upper ring down (note currents are in same direction) is $F_m = \frac{4\pi b}{c^2 h} I^2$ or

$$F_m = \frac{4\pi b}{c^2 h}\left(2\pi f \mathcal{E}_0 C_2\right)^2 \sin^2 2\pi f t$$

The time-average of $\sin^2 2\pi f t$ is simply $\frac{1}{2}$ *
Hence the average force is

$$\overline{F}_m = \frac{8\pi^3 b f^2 \mathcal{E}_0^2 C_2^2}{hc^2}$$

In the capacitor at the left the electric field

* Easily proved from fact that $\overline{\sin^2(\ )} = \overline{\cos^2(\ )}$ and $\sin^2 + \cos^2 = 1$.

strength is: $E = \frac{\mathcal{E}_0 \cos 2\pi f t}{s}$. The downward force on the upper plate is $\frac{E^2}{8\pi} \times \text{area}$, or

$$F_e = \frac{E^2}{8\pi}\cdot \pi a^2 = \frac{\mathcal{E}_0^2 a^2}{8 s^2}\cos^2 2\pi f t$$

The time average of $\cos^2 2\pi f t$ is $\frac{1}{2}$, so

$$\overline{F}_e = \frac{a^2 \mathcal{E}_0^2}{16 s^2}$$

This can be expressed in terms of the capacitance C_1, which is $\frac{\pi a^2}{4\pi s}$ or $\frac{a^2}{4s}$. Substituting for s:

$$\overline{F}_e = \frac{\mathcal{E}_0^2 C_1^2}{a^2}$$

When the forces are balanced (which might be brought about by varying C_2), we have $\overline{F}_e = \overline{F}_m$:

$$\frac{\mathcal{E}_0^2 C_1^2}{a^2} = \frac{8\pi^3 b f^2 \mathcal{E}_0^2 C_2^2}{hc^2}$$

Solving for c:

$$c = (2\pi)^{\frac{3}{2}}\, a \left(\frac{b}{h}\right)^{\frac{1}{2}} f \left(\frac{C_2}{C_1}\right)$$

CHAPTER 8

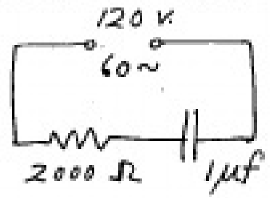
8.1 For a 60-watt, 120-volt bulb the normal current is $I = \frac{P}{V} = \frac{60}{120} = 0.5$ amp. The resistance of the filament is $R = \frac{120}{0.5} = 240$ ohms. We want to have the same current, 0.5 amp, when the bulb is connected in series with a reactance ωL, across 240 volts:

$$0.5 = |I| = \frac{V}{|Z|} = \frac{V}{\sqrt{R^2 + (\omega L)^2}} \qquad 0.5 = \frac{240}{\sqrt{(240)^2 + (\omega L)^2}}$$

$$(\omega L)^2 = 480^2 - 240^2 \qquad \omega L = 240\sqrt{3} = 415 \text{ ohms reactance.}$$

$$\omega = 2\pi f = 2\pi \times 60 = 377 \qquad L = \frac{415}{377} = 1.10 \text{ henry}$$

8.2 (a) $Z = R - \frac{i}{\omega C} = 2000 - \frac{i}{377 \times 10^{-6}}$



$$= 2000 - 2650\, i \text{ ohms}$$

The absolute value of the impedance $|Z|$ is: $|Z| = \sqrt{2000^2 + 2650^2} = 3320$ ohms

(b) The rms current is simply $\frac{V}{|Z|}$ where V is the rms voltage, in this case 120 volts: $I = \frac{120}{3320} = 0.036$ amp.

(c) The power dissipated in the circuit is I^2R:

$$P = I^2R = (0.036)^2 \times 2000 = 2.59 \text{ watts.}$$

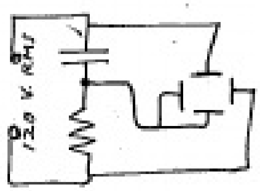
(d) A voltmeter connected across the resistor will read:

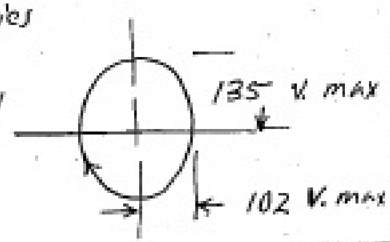
$$V_R = IR = 0.036 \times 2000 = 72 \text{ volts} \quad \text{(rms)}$$

A voltmeter connected across the capacitor will read:

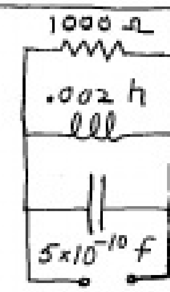
$$V_C = \frac{I}{\omega C} = 0.036 \times 2650 = 95.5 \text{ volts} \quad \text{(rms)}$$

If plates are connected as shown and if vertical and horizontal deflection sensitivities are equal, beam will trace elliptical pattern, in the sense indicated.





8.3 $\frac{1}{Z} = Y = \frac{1}{R} - \frac{i}{\omega L} + i\omega C$



For 10 kc, $\omega = 6.28 \times 10^4$

$\omega L = 125.6 \qquad \omega C = 3.14 \times 10^{-5}$

$$\frac{1}{Z} = \frac{1}{1000} - \frac{i}{125.6} + i\, 3.14 \times 10^{-5}$$
$$= 10^{-3} - i\,(7.96 \times 10^{-3} - 3.14 \times 10^{-5})$$
$$= 10^{-3} - 7.93 \times 10^{-3}\, i$$

$$Z = \frac{1}{10^{-3}(1 - 7.93i)} = \frac{10^3(1 + 7.93i)}{1 + (7.93)^2} = 15.7 + 125i$$

For 10 Mc, $\omega = 6.28 \times 10^7 \quad \omega L = 1.256 \times 10^5 \quad \omega C = 3.14 \times 10^{-2}$

$$\frac{1}{Z} = \frac{1}{1000} - \frac{i}{1.256 \times 10^5} + 3.14 \times 10^{-2}\, i$$
$$= 10^{-3} + i\,(-8 \times 10^{-6} + 3.14 \times 10^{-2}) = 10^{-3} + 3.14 \times 10^{-2}\, i$$

$$Z = \frac{1}{10^{-3}(1 + 31.4i)} = \frac{10^3(1 - 31.4i)}{1 + (31.4)^2} = 1.01 - 31.8\, i$$

$|Z|$ is greatest at resonance, since $Z = \frac{1}{\frac{1}{R} + i(\omega C - \frac{1}{\omega L})}$ and $|Z| = \frac{R}{\sqrt{1 + R^2(\omega C - \frac{1}{\omega L})^2}}$. At resonance, $\omega L = \frac{1}{\omega C}$.

$$\omega_{res} = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{2 \times 10^{-3} \times 5 \times 10^{-10}}} = \sqrt{10^{12}} = 10^6 \text{ sec}^{-1}$$

$$f = \frac{\omega}{2\pi} = 159 \text{ kc/sec} \qquad |Z|_{max} = R = 1000 \text{ ohms}$$

8.4 For the series RLC circuit we had Eq. 2:

$$\frac{d^2V}{dt^2} + \frac{R}{L}\frac{dV}{dt} + \frac{1}{LC}V = 0 \qquad (2)$$

In the parallel circuit with currents defined as indicated, and with V defined as positive when the upper line is positive, we have: $Q = CV \quad I_1 = -\frac{dQ}{dt} = -C\frac{dV}{dt}$



$V = R'(I_1 + I_2)$ and $V = -L\frac{dI_2}{dt}$

$$\frac{dV}{dt} = R'\left(\frac{dI_1}{dt} + \frac{dI_2}{dt}\right) = -R'C\frac{d^2V}{dt^2} - \frac{R'V}{L} \qquad \text{Rearranging,}$$

this becomes:

$$\frac{d^2V}{dt^2} + \frac{1}{R'C}\frac{dV}{dt} + \frac{1}{LC}V = 0$$

Comparing this with Eq. 2, we observe that the equations are identical if $\frac{1}{R'C} = \frac{R}{L}$, or $R' = \frac{L}{RC}$.

Using the same trial solution (Eqs. 3, 4, 5) and substituting as in Eq. 6, we get in place of (7) and (8):

$$2\alpha\omega - \frac{\omega}{CR'} = 0 \quad \text{and} \quad \alpha^2 - \omega^2 - \frac{\alpha}{CR'} + \frac{1}{LC} = 0$$

giving $\alpha = \frac{1}{2CR'}$ and $\omega^2 = \frac{1}{LC} - \frac{1}{4C^2R'^2}$

The two circuits will have the same Q if $R' = L/RC$.

8.5 $V = Ae^{-\alpha t}\cos\omega t \qquad I = AC\omega(\sin\omega t + \frac{\alpha}{\omega}\cos\omega t)e^{-\alpha t}$

At $t=0$, $V = A \quad I = AC\alpha$ Energy in circuit $= \frac{1}{2}CV^2 + \frac{1}{2}LI^2 = \frac{1}{2}CA^2 + \frac{1}{2}LA^2C^2\alpha^2 \approx \frac{1}{2}CA^2$, since we are neglecting terms proportional to α^2.

At $t = \frac{\pi}{2\omega}$, $V = 0 \quad I = AC\omega\, e^{-\frac{\alpha\pi}{2\omega}}$ Energy in circuit $= \frac{1}{2}LI^2 = \frac{1}{2}LA^2C^2\omega^2 e^{-\frac{\alpha\pi}{\omega}}$

To first order in α, $\omega^2 \approx \frac{1}{LC}$, and expanding the exponential, $e^{-\frac{\alpha\pi}{\omega}} = 1 - \frac{\alpha\pi}{\omega}$ + higher-order terms.

Hence the decrease in energy in the circuit is:

$$\frac{1}{2}CA^2 - \frac{1}{2}\frac{LA^2C^2}{LC}\left(1 - \frac{\alpha\pi}{\omega}\right) = \frac{1}{2}A^2C\frac{\alpha\pi}{\omega}$$

which, since $\alpha = \frac{R}{2L}$ and $\omega^2 = \frac{1}{LC}$ can be written:

$$\text{decrease in energy stored} = \frac{\pi}{4}A^2RC^2\omega$$

The energy dissipated in the resistor is $\int_0^{\frac{\pi}{2\omega}} I^2R\,dt$

$$= R\int_0^{\frac{\pi}{2\omega}} A^2C^2\omega^2\sin^2\omega t\,dt = RA^2C^2\omega\int_0^{\pi/2}\sin^2\theta\,d\theta = \frac{\pi}{4}RA^2C^2\omega$$

(The slight decrease in I during this interval, owing to the factor $e^{-\alpha t}$, can be ignored in computing the first-order magnitude of the energy dissipated.) The exact calculation of the decrease in stored energy and the dissipation in the resistor is possible, of course; and must give exact agreement. It is merely more tedious.

8.6 $V(t) = Ae^{-\beta_1 t} + Be^{-\beta_2 t}$ was said to be the general solution in the over-damped case. To verify this, we try the solution $V = Ae^{-\beta t}$ for Eq. 2:

$$\frac{dV}{dt} = -A\beta e^{-\beta t} \qquad \frac{d^2V}{dt^2} = A\beta^2 e^{-\beta t}$$

Substituting in $\frac{d^2V}{dt^2} + \frac{R}{L}\frac{dV}{dt} + \frac{1}{LC}V = 0$ we have:

$$A\beta^2e^{-\beta t} - \frac{R}{L}A\beta e^{-\beta t} + \frac{A}{LC}e^{-\beta t} = 0 \quad \text{or} \quad \beta^2 - \frac{R}{L}\beta + \frac{1}{LC} = 0$$

The roots of this quadratic equation are:

$$\beta = \frac{1}{2}\left(\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}\right) = \frac{R}{2L}\left(1 \pm \sqrt{1 - \frac{4L}{R^2C}}\right)$$

The roots are real if $R \geqslant 2\sqrt{\frac{L}{C}}$ Notice that both roots are positive. Call them β_1 and β_2. With $R = 600\,\Omega$, $L = 10^{-4}$ henrys $C = 10^{-8}$ farads,

$$\frac{4L}{R^2C} = \frac{4\times10^{-4}}{36\times10^4\times10^{-8}} = \frac{1}{9} \qquad \frac{R}{2L} = \frac{600}{2\times10^{-4}} = 3\times10^6$$

$$\beta_1 = 3\times10^6\left(1+\sqrt{\tfrac{8}{9}}\right) = 3\times10^6\times1.943 = 5.84\times10^6$$

$$\beta_2 = 3\times10^6\left(1-\sqrt{\tfrac{8}{9}}\right) = 3\times10^6\times0.057 = 0.171\times10^6$$

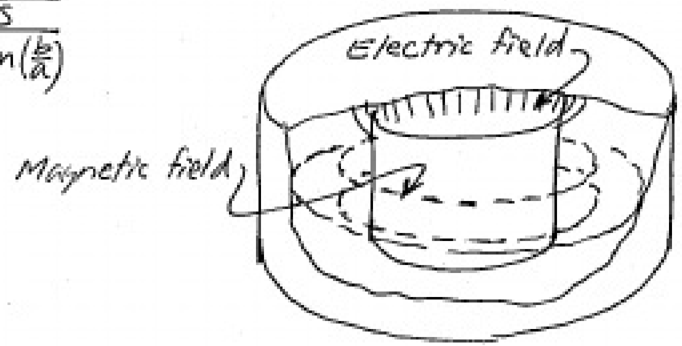
The initial condition is $\frac{dV}{dt} = 0$ at $t = 0$. Hence $-\beta_1A - \beta_2B = 0$, or $\frac{B}{A} = -\frac{\beta_1}{\beta_2} = 34$. The term $Be^{-\beta_2 t}$ is the *slowly* decaying term, and after a microsecond or so, it is practically all that is left.

8.7 From Eq. 7.58, page 251, putting $N=1$, we get the inductance of a single-turn toroid:

$L = \frac{2h}{c^2}\ln\left(\frac{b}{a}\right)$ The capacitance at the gap is:

$$C = \frac{\pi a^2}{4\pi s} = \frac{a^2}{4s} \qquad \omega^2 = \frac{1}{LC} = \frac{c^2}{2h\ln\frac{b}{a}}\frac{4s}{a^2}$$

$$\omega = \frac{c}{a}\sqrt{\frac{2s}{h\ln\left(\frac{b}{a}\right)}}$$



8.8 $V = Ae^{-\alpha t}\cos\omega t \qquad I = AC\omega e^{-\alpha t}(\sin\omega t + \frac{\alpha}{\omega}\cos\omega t)$

with $\alpha = \frac{R}{2L}$ and $\omega^2 = \frac{1}{LC} - \alpha^2$ Energy in circuit

$$= \frac{1}{2}CV^2 + \frac{1}{2}LI^2 = \frac{1}{2}CA^2e^{-2\alpha t}\left[\cos^2\omega t + LC\omega^2\left(\sin\omega t + \frac{\alpha}{\omega}\cos\omega t\right)^2\right]$$

$$= \frac{1}{2}CA^2e^{-2\alpha t}\left[\cos^2\omega t + LC\omega^2\left(\sin^2\omega t + \frac{2\alpha}{\omega}\sin\omega t\cos\omega t + \frac{\alpha^2}{\omega^2}\cos^2\omega t\right)\right]$$

$$= \frac{1}{2}CA^2e^{-2\alpha t}\left[\cos^2\omega t + \sin^2\omega t + LC(-\alpha^2\sin^2\omega t + \alpha^2\cos^2\omega t + 2\alpha\omega\sin\omega t\cos\omega t)\right]$$

$$= \frac{1}{2}CA^2e^{-2\alpha t}\left[1 + LC\left(\alpha^2\cos 2\omega t + \alpha\omega\sin 2\omega t\right)\right]$$

Except for the oscillating part of the factor in brackets, the energy decays exponentially. To dissipate the energy as quickly as possible, we want α to be as large as possible. But the limiting value of α, approaching from the under-damped side, corresponds to $R = 2\sqrt{\frac{L}{C}}$ or $\alpha = \frac{R}{2L} = \frac{1}{\sqrt{LC}}$. In that limit, $\omega = 0$ and the expression above reduces to $CA^2e^{-2\alpha t} = CA^2e^{-\frac{R}{L}t}$.

Look now at the over-damped case, using the results from Problem 8.6. We found $V = Ae^{-\beta_1 t} + Be^{-\beta_2 t}$

with $\beta_1 = \frac{R}{2L}\left(1 + \sqrt{1 - \frac{4L}{R^2C}}\right)$ and $\beta_2 = \frac{R}{2L}\left(1 - \sqrt{1 - \frac{4L}{R^2C}}\right)$

Evidently the root β_2 is necessarily less than $\frac{R}{2L}$, so there will be a term in the energy decaying like $e^{-2\beta_2 t}$ which will decay more slowly than $e^{-\frac{R}{L}t}$. Hence the over-damped case exhibits slower decay than the case of critical damping, as does the underdamped case.

8.9 From Eq. 10, $\omega^2 = \frac{1}{LC} - \alpha\frac{R}{L} + \alpha^2 = \frac{1}{LC} - \frac{R^2}{4L^2}$

Define: $\omega_0 \equiv \frac{1}{\sqrt{LC}} \qquad Q = \omega\frac{L}{R}$, as explained before Eq. 14. Then $\omega^2 = \frac{1}{LC} - \frac{R^2}{4L^2} = \omega_0^2 - \frac{\omega^2}{4Q^2}$

Hence $\omega^2 = \frac{\omega_0^2}{1 + \frac{1}{4Q^2}}$

$$\omega = \omega_0\left[1 + \frac{1}{4Q^2}\right]^{-\frac{1}{2}} \simeq \omega_0\left[1 - \frac{1}{2}\left(\frac{1}{4Q^2}\right)\right] \quad \text{if } \frac{1}{4Q^2} \ll 1.$$

In our case, with $Q = 1000$, the approximation is very good.

The fraction frequency difference is

$$\frac{\omega_0 - \omega}{\omega_0} = \frac{1}{8Q^2} = \frac{1}{8\times10^6} = 1.2\times10^{-7}$$

8.10 For the long single-turn solenoid $B = \frac{4\pi}{c}\frac{I}{b}$ and $\Phi = \pi a^2 B = \frac{4\pi^2a^2I}{cb}$. Hence $L = \frac{4\pi^2a^2}{c^2b}$

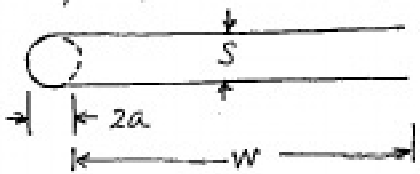
$$C = \frac{bw}{4\pi s} \qquad \omega = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\frac{bw}{4\pi s}\cdot\frac{4\pi^2a^2}{c^2b}}} = \frac{c}{a}\sqrt{\frac{s}{\pi w}}$$

Estimating $a = 0.5$ cm and $s/w = 1/2$ for the magnetron cavity, we have $\omega = \frac{3\times10^{10}}{.5}\sqrt{\frac{1}{2\pi}} = 2.4\times10^{10}$ rad/sec

This corresponds to a wavelength $\lambda = \frac{2\pi c}{\omega} \simeq 8$ cm. (The actual device operated at about 10 cm wavelength.)

When we use our formula to estimate the resonant

frequency of the shape (d), we are assuming that the magnetic field is confined to the imaginary cylinder of diameter 2a. Of course it isn't — it spreads



out between the flat sheets. The actual inductance is hence larger than our formula assumes and the actual frequency will be somewhat lower. [In fact, in the limit $a \ll w$, the resonant frequency is given quite accurately by setting w equal to a quarter-wavelength: $\lambda = 4w$ or $\omega = \frac{\pi}{2}\frac{c}{w}$.]

CHAPTER 9

9.1 1 farad = 9×10^{11} cm (p.97)

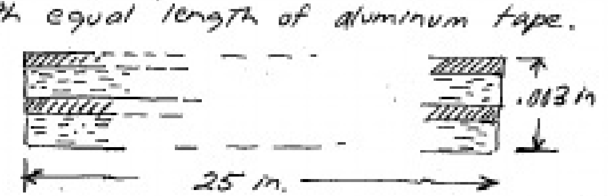
$$C \text{ (farads)} = \frac{1}{9\times10^{11}} \times \frac{\text{area} \times \text{dielectric constant}}{4\pi \times \text{plate separation}}$$

For $C = 5\times10^{-8}$, $\epsilon = 2.3$, plate separation = thickness of polyethylene tape = 0.001 inch = 0.00254 cm,

$$\text{area required} = \frac{9\times10^{11}\times4\pi\times5\times10^{-8}\times2.54\times10^{-3}}{2.3} = 625 \text{ cm}^2$$

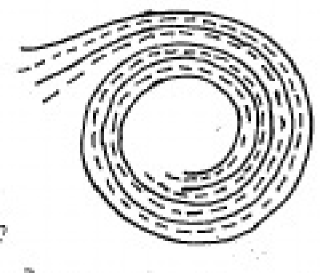
width of aluminum tape = 2 in = 5.08 cm

total length of dielectric tape required = $\frac{625}{5.08}$ = 123 cm with equal length of aluminum tape. Arrange as follows:



123 cm = 48.5 in, call it 50 in.

Roll up

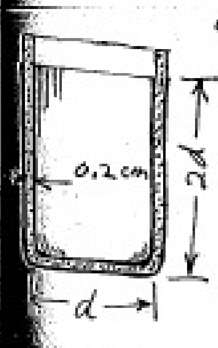


Area of cross-section normal to axis = 25 × .003 = .075 in²

If no hole in center, diameter of roll, d, must be such that $\frac{\pi}{4}d^2 = .075$ or $d = .31$ in.

Allowing for core on which to wind, overall diameter may be somewhat greater.

9.2 Assume height = 2d (result will depend somewhat on proportions assumed).



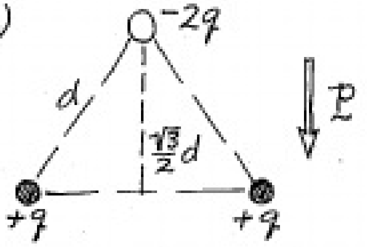
1 liter = 10^3 cm³ = $2d \times \frac{\pi}{4}d^2$, or $d = 8.6$ cm

area of capacitor = $\pi d \times 2d + \frac{\pi}{4}d^2 = \frac{9}{4}\pi d^2$ = 522 cm²

$$C = \frac{522\times4}{4\pi\times0.2} = 830 \text{ cm}$$

This is the capacitance of a sphere 830 cm radius; or about 54 feet diameter.

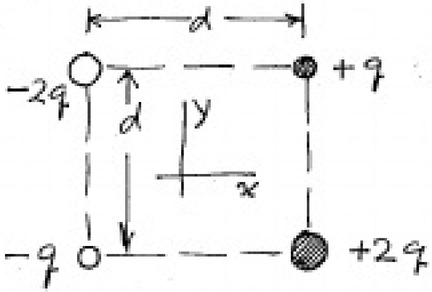
9.3 (a)



$p = 2q\frac{\sqrt{3}}{2}d = \sqrt{3}\,qd$

(b) $p = 0$

(c)



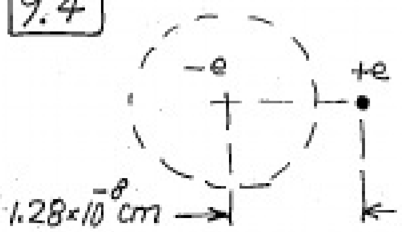
$p_x = +3qd$

$p_y = -qd$

$p = \sqrt{10}\,qd = 3.16\,qd$

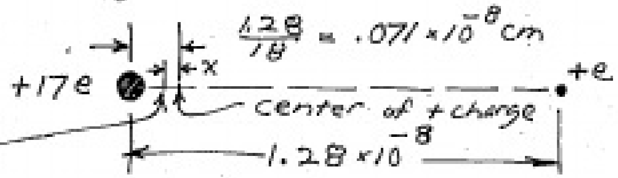


9.4



$p = 1.28\times10^{-8}$ cm $\times 4.8\times10^{-10}$ esu $= 6.15\times10^{-18}$ esu-cm, which is about six times the actual dipole moment, 1.03×10^{-18}.

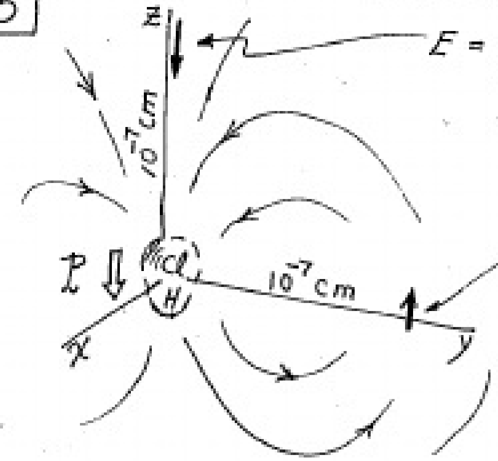
By the "center" of a charge distribution we mean the point with respect to which, as origin, $\int \Sigma \rho\, d\vec{r} = 0$. Considering the positive charges by themselves, the center lies $\frac{1}{18}(1.28\times10^{-8}\text{ cm})$ from the Cl nucleus:



center of negative charge distribution, total charge $-18e$, must lie to left of center of + charge by distance x such that $18ex = 1.03\times10^{-18}$, or

$$x = \frac{1.03\times10^{-18}}{18\times4.8\times10^{-10}} = .0119\times10^{-8}\text{ cm}$$

9.5



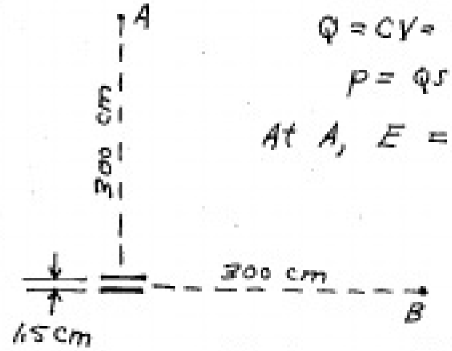
$E = \frac{2p}{z^3} = \frac{2\times1.03\times10^{-18}}{(10^{-7})^3}$

$= 2.06\times10^{3}\ \frac{\text{statvolts}}{\text{cm}}$

$E = \frac{p}{y^3}$

$= 1.03\times10^{3}\ \frac{\text{statvolts}}{\text{cm}}$

9.6



$Q = CV = 250\times6 = 1500$ esu

$p = Qs = 1500\times1.5 = 2250$ esu-cm

At A, $E = \frac{2p}{r^3} = \frac{2\times2250}{(300)^3}$

$= 1.67\times10^{-4}$ statvolts/cm

At B, $E = \frac{p}{r^3} = 0.833\times10^{-4}$ statvolts/cm

9.7 The time constant was $\tau = RC = \frac{\rho s}{A}\cdot\frac{A}{4\pi s} = \frac{\rho}{4\pi}$

With dielectric constant ϵ, C is increased by that factor, so $\tau = \frac{\rho\epsilon}{4\pi}$.

If C, per cm², is 10^{-6} farad, this implies a dielectric thickness s such that

$10^{-6}\times9\times10^{11} = \frac{\epsilon}{4\pi s}$ or $s = \frac{3}{4\pi\times9\times10^{5}} = 2.6\times10^{-7}$ cm

$\tau = RC = 1000$ ohms $\times 10^{-6}$ farads $= 10^{-3}$ sec (independent of area, since $C \sim$ area, $R \sim$ area^{-1}).

$\rho = \frac{1000}{2.6\times10^{-7}} \approx 4\times10^{9}$ ohm-cm.

9.8 $p = 1.84 \times 10^{-18}$ (Fig. 9.16, Water)

volume $= \frac{4\pi}{3}(0.1)^3 = 4\times 10^{-3}\ \text{cm}^3$ Mol. Wt. = 18

mass $= 4\times10^{-3}$ gm (density of water = 1 gm/cm^3)

No. of molecules $= \frac{4\times10^{-3}}{18}\times 6\times10^{23} = 1.34\times10^{20}$

$p = 1.34\times10^{20}\times1.84\times10^{-18} = 246$ esu-cm

E at 10 cm distance on axis $= \frac{2p}{10^3} = \frac{2\times246}{10^3}$
$= 0.49$ statvolts/cm

E at surface of drop $= \frac{2p}{(0.1)^3} = 4.9\times10^5$ statvolts/cm

9.9 Refer to Sect. 9.11 Eq. 50 gives the polarization of the dielectric sphere:

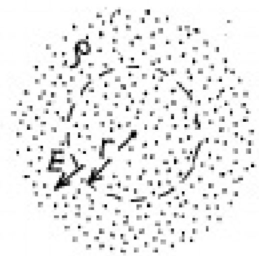
$P = \frac{3}{4\pi}\frac{\epsilon-1}{\epsilon+2}E_0$. In the limit $\epsilon\to\infty$, this becomes $P = \frac{3}{4\pi}E_0$. The field strength inside the dielectric sphere, $E = \frac{3}{2+\epsilon}E_0$ (Eq. 49) goes to zero as $\epsilon\to\infty$. In the limit the sphere becomes an equipotential, which is correct for a conducting sphere. $p = \text{volume}\times P = \frac{4\pi}{3}a^3\times\frac{3}{4\pi}E_0 = a^3E_0$

Hence the polarizability α, defined by $p=\alpha E_0$, is just a^3 for the conducting sphere of radius a.
$\alpha_H = 0.66\times10^{-24}\ \text{cm}^3$ (Table 9.2) A conducting sphere of equal polarizability has a radius
$(0.66\times10^{-24})^{1/3} = 0.87\times10^{-8}$ cm.

9.10 $P = \frac{3}{4\pi}\frac{\epsilon-1}{\epsilon+2}E_0$ For $\epsilon = 81$, $P = \frac{3}{4\pi}\frac{80}{83}E_0$

For $\epsilon=\infty$, $P = \frac{3}{4\pi}E_0$ Dipole moment of water drop is $\frac{80}{83}$ as large as, or about 3.5% less than, that of conducting sphere of the same diameter.

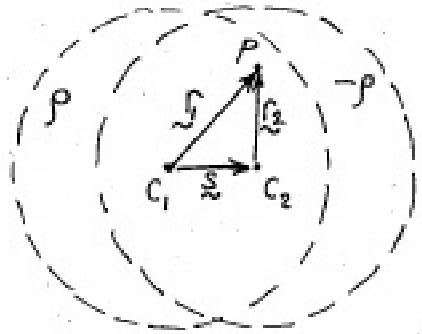
9.11 (a) Apply Gauss's law to sphere of radius r: Charge enclosed $= \frac{4\pi}{3}r^3\rho$;



$4\pi r^2E = 4\pi q = 4\pi\left(\frac{4\pi}{3}r^3\rho\right)$

$E = \frac{4\pi}{3}\rho r$

(b)



Let $\mathbf{s}$ be the vector displacement from C_1 to C_2. At the point P,

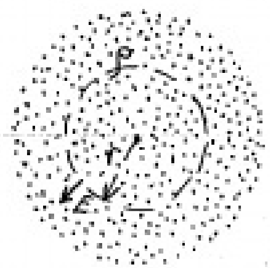
$\mathbf{E}_1 = \frac{4\pi}{3}\rho\,\mathbf{r}_1$ and

$\mathbf{E}_2 = \frac{4\pi}{3}(-\rho)\,\mathbf{r}_2$

The total field is thus $\mathbf{E} = \mathbf{E}_1+\mathbf{E}_2 = \frac{4\pi}{3}\rho\,(\mathbf{r}_1-\mathbf{r}_2)$
But $\mathbf{r}_1-\mathbf{r}_2 = \mathbf{s}$ Hence $\mathbf{E} = \frac{4\pi\rho}{3}\mathbf{s}$
This holds for any point inside both distributions. Note that $-\mathbf{s}$ is the displacement of the positive charge distribution with respect to the negative, so that the volume polarization is $\mathbf{P} = -\rho\mathbf{s}$. Therefore,

$\mathbf{E} = \frac{4\pi\rho\mathbf{s}}{3} = -\frac{4\pi}{3}\mathbf{P}$, in agreement with Eq. 49.

(c) For the cylindrical distribution, Gauss's law applied to unit length of cylinder is:



$2\pi r E = 4\pi(\pi r^2\rho)$, or $E = 2\pi r\rho$

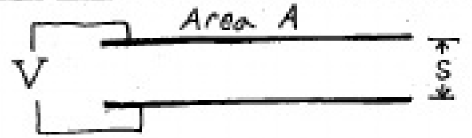
As in (b):

$\mathbf{E}_1 = 2\pi\rho\,\mathbf{r}_1$

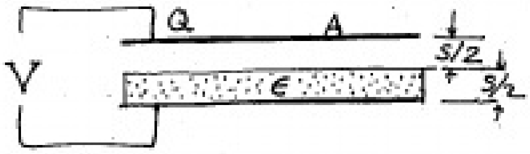
$\mathbf{E}_2 = 2\pi(-\rho)\,\mathbf{r}_2$

Thus $\mathbf{E} = \mathbf{E}_1+\mathbf{E}_2 = 2\pi\rho(\mathbf{r}_1-\mathbf{r}_2) = 2\pi\rho\mathbf{s}$ In this case also, $\mathbf{P} = -\rho\mathbf{s}$ so we obtain, for the field inside a cylinder with uniform transverse polarization: $\mathbf{E} = -2\pi\mathbf{P}$

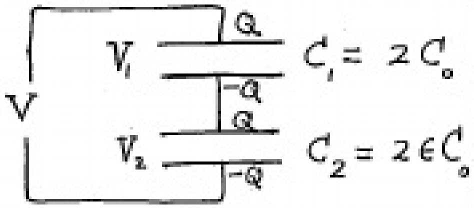
9.12



$$C_0 = \frac{A}{4\pi S}$$



This is equivalent to two capacitors, C_1 and C_2 in series.

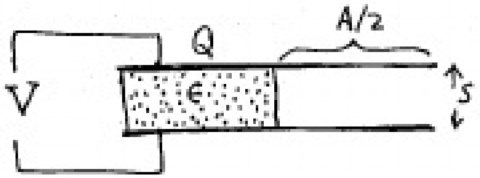


$$V_1 = \frac{Q}{C_1} = \frac{Q}{2C_0}$$

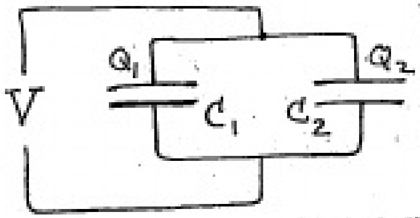
$$V_2 = \frac{Q}{C_2} = \frac{Q}{2\epsilon C_0} \qquad V = V_1 + V_2 = \frac{Q}{2C_0}\left(1 + \frac{1}{\epsilon}\right)$$

The capacitance of the combination is:

$$C = \frac{Q}{V} = \frac{2C_0}{1+\frac{1}{\epsilon}} = \frac{2\epsilon}{\epsilon+1} C_0$$



This is equivalent to two capacitors C_1 and C_2 in parallel.



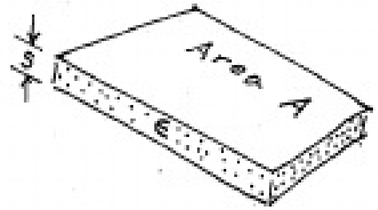
$$C_1 = \frac{\epsilon C_0}{2} \qquad C_2 = \frac{C_0}{2}$$

$$Q_1 = C_1 V = \frac{\epsilon}{2} C_0 V \qquad Q_2 = C_2 V = \frac{C_0}{2} V$$

The capacitance of the combination is:

$$C = \frac{Q}{V} = \frac{Q_1 + Q_2}{V} = \frac{\epsilon + 1}{2} C_0$$

13 Electric field E in



pacitor is: $E = \frac{V}{S}$

$C = \frac{A\epsilon}{4\pi S}$ Energy stored

: $U = \frac{1}{2}CV^2 = \frac{1}{2}\frac{A\epsilon}{4\pi S}E^2 S^2$

$= \frac{1}{8\pi}\epsilon E^2 AS = \frac{\epsilon E^2}{8\pi} \times$ volume of dielectric

(Eq. 74, p. 338)

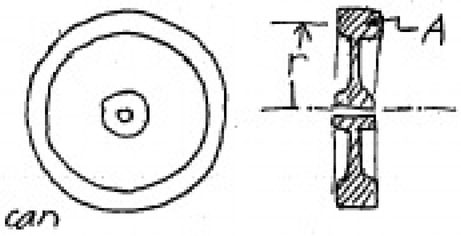
ith $E = 10^5$ volts/cm $= \frac{10^5}{300}$ statvolts/cm, and $\epsilon = 2.3$

U/volume $= \frac{1}{8\pi}\left(\frac{10^5}{300}\right)^2 \times 2.3 = 1.0 \times 10^4$ ergs/cm^3

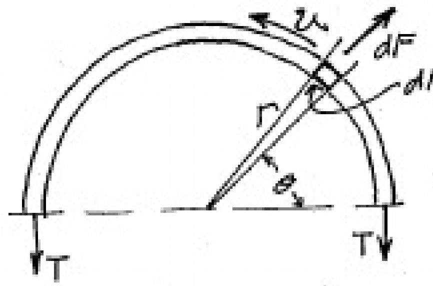
density is 1 gm/cm^3, this is equivalent to 10^7 ergs/kg
1 joule/kilogram.

A typical automobile storage battery might have aracteristics like this: 12 volts; 50 ampere-hours pacity; 35 lbs weight. Since 1 watt-second is one le, the energy obtainable from the battery is
$12 \times 50 \times 3600 = 2.16 \times 10^6$ joules, for a mass
$35/2.2 = 16$ kilograms, or 135 000 joules/kilogram

Energy stored in a flywheel is simply kinetic ergy $\frac{1}{2}mv^2$. For a 1 kg mass, the energy in les is $\frac{1}{2}v^2$ where v is a suitable mean speed in $\frac{m}{sec}$. $v = 200$ m/sec, about 600 ft/sec or 400 miles/hr, we have (200^2) or 20 000 joules/kilogram. (We know some wheels that fast — consider the land speed record.) Actually speed of a flywheel is limited only by its strength, the limit turns out to be independent of size. sider a flywheel with its mass concentrated in rim of mean radius r cross-section area A, de of material with sity ρ in gm/cm^3. king in CGS units, we can



compute, for a given rim speed v, the tensile stress in the rim



$dm = \rho A r d\theta$

$$dF = dm \cdot \frac{v^2}{r} = \rho A v^2 d\theta$$

$$2T = \int_{\theta=0}^{\theta=\pi} \sin\theta \, dF = \rho A v^2 \int_0^{\pi} \sin\theta \, d\theta$$

$2T = 2\rho A v^2$. The tensile stress in the rim is T/A or ρv^2. If S is the limiting tensile strength of the material, in dynes/cm², the wheel will be ready to burst when $v^2 = \frac{S}{\rho}$. A reasonable figure for steel is $S = 10^{10}$ dynes/cm², with $\rho = 8$ gm/cm³. In that case

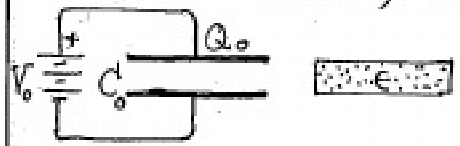
$$v^2_{max} = \frac{10^{10}}{8} \quad \text{or} \quad v_{max} \simeq 3.5 \times 10^4 \text{ cm/sec}$$

or 350 meters/sec, the kinetic energy stored is $\frac{1}{2}(350^2)$ or 60 000 joules/kilogram.

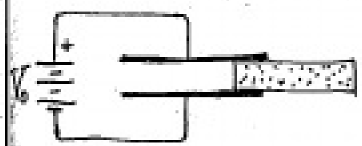
(Other alloys and materials exist with higher strength/density ratios. But also, there are batteries that weigh less, per joule stored, than the conventional lead storage battery. A kilogram of gasoline yields 5×10^7 joules of energy when burned.)

As an energy reservoir, the capacitor by this criterion appears some 10^5 times worse than a battery or a flywheel. But the capacitor can be made to deliver its entire energy in a fraction of a microsecond.

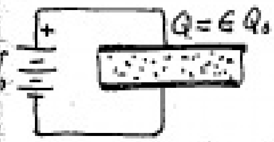
9.14 Process I: dielectric slab moves into capacitor while plates are connected to battery of constant emf V_0.



(a) $Q_0 = C_0 V_0$. Stored energy, $U = \frac{1}{2} C_0 V_0^2 = \frac{1}{2} Q_0 V_0$



(b) C increases as dielectric moves in; V remains constant at V_0. Q increases, as charge flows from battery.



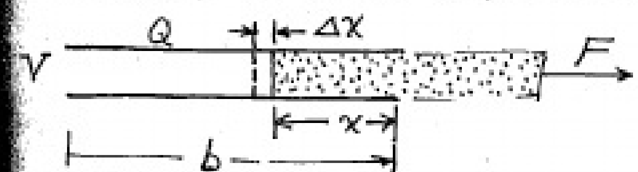
(c) $C = \epsilon C_0$. Stored energy is now: $U = \frac{1}{2} C V_0^2 = \frac{1}{2} \epsilon C_0 V_0^2$

Increase in stored energy: $\frac{1}{2}(\epsilon - 1) C_0 V_0^2$

Energy drawn from battery $= V_0(Q - Q_0) = (\epsilon - 1) C_0 V_0^2$

The missing energy, $(\epsilon-1)C_0V_0^2 - \frac{1}{2}(\epsilon-1)C_0V_0^2 = \frac{1}{2}(\epsilon-1)C_0V_0^2$ is work done on something else, namely, whatever was holding the slab of dielectric as it moved into the capacitor. The sign shows that the slab is sucked into the capacitor, so if it is to move in without accelerating and acquiring kinetic energy, a steady outward force must be applied to it by another agent.

Consider a small displacement, Δx, of the slab.



$$C = C_0\left[\frac{\epsilon x}{b} + \frac{b-x}{b}\right] = \frac{C_0(b + \epsilon x - x)}{b}$$

Let ΔU be the change in stored energy as the

slab moves inward by Δx. Let ΔW be the work done by the battery. Then <u>if V is constant</u>, $V = V_0$, $\Delta Q = V_0 \frac{dC}{dx}\Delta x$

$$\Delta U = \Delta\left(\tfrac{1}{2}CV^2\right) = \tfrac{1}{2}V_0^2 \frac{dC}{dx}\Delta x$$

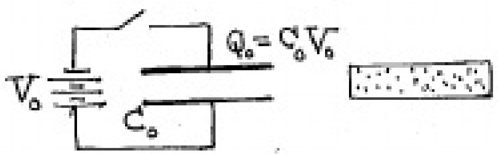
$$\Delta W = V_0 \Delta Q = V_0^2 \frac{dC}{dx}\Delta x$$

Work done against force $F = F\Delta x = \Delta W - \Delta U$

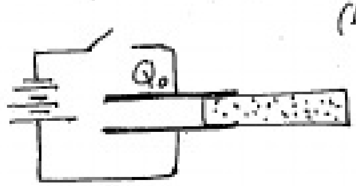
$$F\Delta x = V_0^2 \frac{dC}{dx}\left(1-\tfrac{1}{2}\right)\Delta x$$

$$F = \tfrac{1}{2}V_0^2 \frac{dC}{dx} = \tfrac{1}{2}V_0^2 \frac{\epsilon - 1}{b} C_0$$

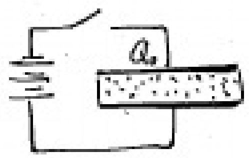
Process II: dielectric slab moves into capacitor after plates have been charged to potential difference V_0 and then disconnected from battery.



(a) $U = \tfrac{1}{2}C_0V_0^2 = \tfrac{1}{2}Q_0V_0$



(b) C increases. Q remains constant at Q_0. V must decrease.



(c) $C = \epsilon C_0$ $\quad V = \frac{Q_0}{C} = \frac{V_0}{\epsilon}$

$$U = \tfrac{1}{2}Q_0V = \tfrac{1}{2}\frac{C_0V_0^2}{\epsilon}$$

$\Delta U = \tfrac{1}{2}C_0V_0^2\left(\frac{1}{\epsilon} - 1\right)$ Stored energy has <u>decreased</u> in this case, but no work was done by battery, obviously, so this decrease is accounted for by work done against external

force F. Here also, slab is sucked into capacitor. To compute force consider as before a small displacement Δx:

$$\Delta U = \Delta\left(\tfrac{1}{2}CV^2\right) = \Delta\left(\tfrac{1}{2}\frac{Q_0^2}{C}\right) = \frac{Q_0^2}{2}\Delta\left(\frac{1}{C}\right),$$

since Q is constant at the value Q_0.

$$\Delta\left(\frac{1}{C}\right) = -\frac{1}{C^2}\frac{dC}{dx}\Delta x \qquad \text{Hence:}$$

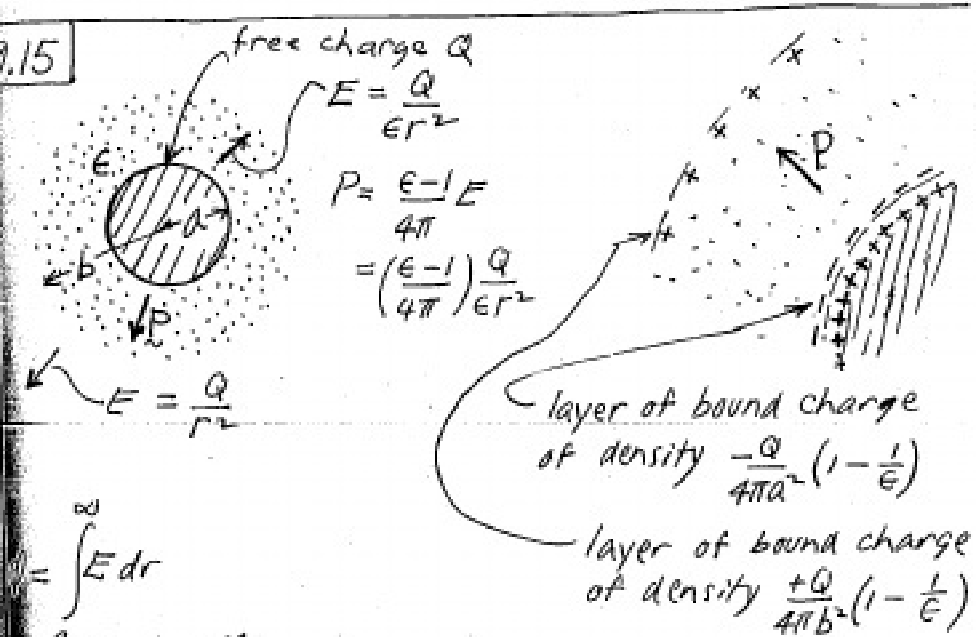
$$\Delta U = -\frac{Q_0^2}{2C^2}\frac{dC}{dx}\Delta x = -\frac{V^2}{2}\frac{dC}{dx}\Delta x$$

In this case, $F\Delta x = -\Delta U = \frac{V^2}{2}\frac{dC}{dx}\Delta x$

or $F = \frac{V^2}{2}\frac{dC}{dx} = \frac{1}{2}V^2\frac{(\epsilon-1)}{b}C_0$

Notice that for given V, the force is exactly the same as in Process I.

9.15



$$V = \int_a^\infty E\,dr$$

$$= \int_a^b \frac{Q\,dr}{\epsilon r^2} + \int_b^\infty \frac{Q\,dr}{r^2} = \frac{Q}{\epsilon}\left(\frac{1}{a} - \frac{1}{b}\right) + \frac{Q}{b}$$

9.16 Let ϵ be the dielectric constant of air at atmospheric density. $\epsilon = 1+\delta$, where $\delta \ll 1$. Then if C_0 is the capacitance of the capacitor in vacuum, $C_0(1+\delta)$ is its capacitance when the bell jar, and hence the capacitor, are full of air. $\omega = \frac{1}{\sqrt{LC}}$. Assuming nothing else changes,

$$\frac{\omega_{vac}}{\omega_{atmos}} = \frac{\sqrt{LC_0(1+\delta)}}{\sqrt{LC_0}} = \sqrt{1+\delta} \approx 1+\frac{\delta}{2}, \text{ for } \delta \ll 1.$$

$$\omega_{vac} = 2\pi\,(41.418 \pm 0.002 \text{ Mc/sec})$$

$$\omega_{atmos.} = 2\pi\,(41.405 \pm 0.002 \text{ Mc/sec})$$

If we treat the uncertainty, ± 0.002, as a probable error, the probable error in the difference of the two measured frequencies would be $\sqrt{2}\,(.002)$ or about 0.003. Then

$$\frac{\delta}{2} = \frac{0.013 \pm 0.003}{41.4} \quad \text{from which}$$

$$\epsilon = 1+\delta = 1.00063 \pm 0.00015$$

Some other effects that might contribute to the change in frequency:

(a) a change in dimensions of the helix or the capacitor, or both.

(b) the effect of the magnetic susceptibility of air (See Chap. 10) on the inductance of the helix. (this effect would be quite small, actually ~ of the order of 10^{-8})

9.17 $E = \frac{2p}{r^3}$ $\quad F = eE = \frac{2pe}{r^3}$ $\quad p = 1.84 \times 10^{-18}$ esu-cm

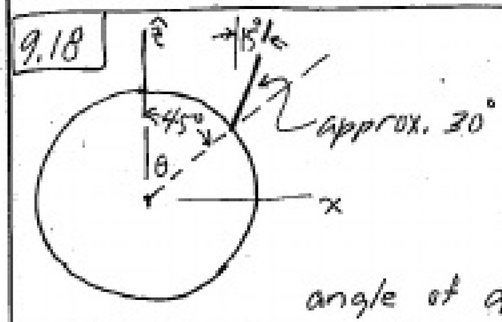


$$\text{work} = \int_{r_1}^{\infty} F\,dr = \int_{r_1}^{\infty} \frac{2pe}{r^3}\,dr = \frac{ep}{r_1^2}$$

$$= \frac{4.8\times 10^{-10} \cdot 1.84\times 10^{-18}}{(1.5\times 10^{-8})^2} = 3.9\times 10^{-12} \text{ erg} = 2.4 \text{ ev.}$$

The negative ion will be near the "H_2" end of H_2O.

9.18



Consider the field line that comes in to the surface of the sphere at a latitude $\theta = 45°$. It appears to make an angle of approximately 30° with the radial direction, or 15° with the z axis. The field is a superposition of a uniform field $E_0\hat{z}$ and the field of a dipole of strength $p = \frac{4\pi}{3}a^3 P$. The volume polarization P is given by Eq. 50, page 329: $P = \frac{3}{4\pi}\left(\frac{\epsilon-1}{\epsilon+2}\right)E_0$. Thus $p = E_0 a^3 \frac{\epsilon-1}{\epsilon+2}$

From Eq. 14, or Fig. 9.5, for $\theta = 45°$,

$$E_x' = \frac{3p\sin\theta\cos\theta}{r^3} = \frac{3p}{2a^3} = \frac{3}{2}E_0\frac{\epsilon-1}{\epsilon+2}$$

$$E_z' = \frac{p(3\cos^2\theta - 1)}{r^3} = \frac{p}{2a^3} = \frac{1}{2}E_0\frac{\epsilon-1}{\epsilon+2}$$

(dipole field)

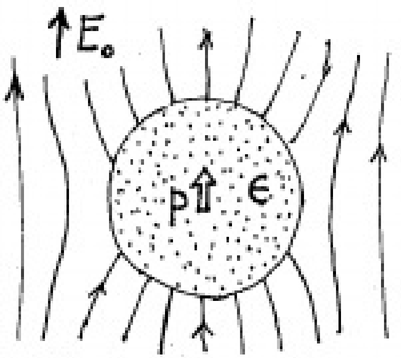
The total field, z-component is

$$E_0 + E_z' = E_0\left(1+\frac{1}{2}\frac{\epsilon-1}{\epsilon+2}\right) = \frac{E_0}{2}\cdot\frac{3\epsilon+3}{\epsilon+2}$$

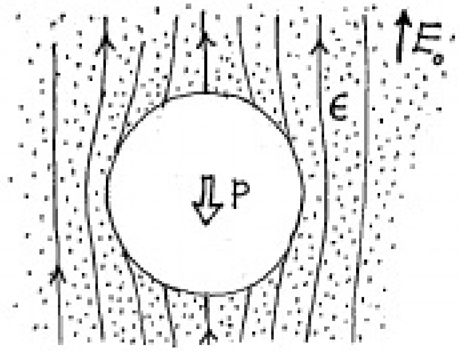
$$\frac{E_x}{E_z} = \frac{3}{2}\frac{\epsilon-1}{\epsilon+2}\cdot\frac{2(\epsilon+2)}{3\epsilon+3} = \frac{\epsilon-1}{\epsilon+1} = \tan 15° = 0.268$$

Solving for ϵ: $\epsilon = \frac{1.268}{0.732} = 1.7$

9.19 The field outside a dielectric sphere is shown in Fig. 9.29. It is composed of the uniform field E_0 and the field E', E' being the field of a dipole p located at the center of the sphere and pointing in the direction of E_0. Inside the dielectric sphere we find the uniform field E_i. If now dielectric and vacuum are interchanged, we would expect, by comparison, something like the field in (b).

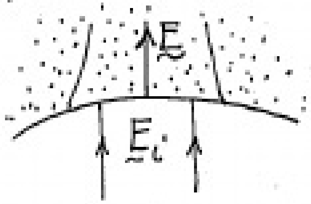


(a) Outside sphere the field is $\underset{\sim}{E} = \underset{\sim}{E}_0 + \underset{\sim}{E}'$, where $\underset{\sim}{E}'$ is field of $\underset{\sim}{p}$ in vacuum. Inside field is uniform: $\underset{\sim}{E} = \underset{\sim}{E}_i$.



(b) Outside sphere the field is $\underset{\sim}{E} = \underset{\sim}{E}_0 + \underset{\sim}{E}'$, where $\underset{\sim}{E}'$ is field of $\underset{\sim}{p}$ in dielectric. Inside field is uniform, $\underset{\sim}{E} = \underset{\sim}{E}_i$.

To find the relation between E_0, p, and E_i in case (b), we examine the relation between the inside and outside fields at the pole and at the equator:



<u>At the pole:</u> $E = E_0 - \dfrac{2p}{\epsilon a^3}$

and the condition on the perpendicular field component at a dielectric boundary requires $\epsilon E = E_i$. Hence

$$\frac{E_i}{\epsilon} = E_0 - \frac{2p}{\epsilon a^3} \qquad (1)$$



<u>At the equator:</u> $E = E_0 + \dfrac{p}{\epsilon a^3}$

and the condition on the parallel field component at a dielectric boundary requires $E = E_i$,

$$E_i = E_0 + \frac{p}{\epsilon a^3} \qquad (2)$$

Eliminating p from (1) and (2) gives:

$$\frac{E_i}{\epsilon} + 2E_i = 3E_0 \quad \text{or} \quad E_i = \frac{3\epsilon}{2\epsilon+1} E_0$$

Comparing this with Eq. 49 we see that it

could have been obtained from Eq. 49 merely by replacing ϵ by $\frac{1}{\epsilon}$. We might have guessed this. It means that the relative field strengths, inside and outside, depend only on the ratio of the dielectric constants of the sphere and the surrounding medium.

With $E_0 = 90$ kV/cm and $\epsilon = 2.24$ we have

$$E_i = 90\,\frac{3 \times 2.24}{4.48+1} = 123 \text{ kV/cm}$$

The field inside the bubble is stronger than the field in the oil. Moreover the dielectric strength of the air in the bubble is probably less than that of the oil; local ionization may be the result.

9.21 Let $\hat{\underset{\sim}{n}}_1$, $\hat{\underset{\sim}{n}}_2$, and $\hat{\underset{\sim}{n}}_3$ be unit vectors in three coplanar directions defined by the molecular symmetry. A field $\underset{\sim}{E}_1 = E_1\hat{\underset{\sim}{n}}_1$ causes a polarization $\underset{\sim}{p} = \alpha E_1 \hat{\underset{\sim}{n}}_1$ ($\underset{\sim}{p}$ must be parallel to $\hat{\underset{\sim}{n}}_1$, if $\underset{\sim}{E}_1$ is, by symmetry) Similarly, a field $\underset{\sim}{E}_2 = E_2\hat{\underset{\sim}{n}}_2$ must cause a polarization $\underset{\sim}{p}_2 = \alpha E_2 \hat{\underset{\sim}{n}}_2$, with the same value of α. Any vector in the plane can be written as $\underset{\sim}{E} = E_1\hat{\underset{\sim}{n}}_1 + E_2\hat{\underset{\sim}{n}}_2$ so that, if superposition holds,



$$\underset{\sim}{p} = \alpha E_1 \hat{\underset{\sim}{n}}_1 + \alpha E_2 \hat{\underset{\sim}{n}}_2 = \alpha(E_1\hat{\underset{\sim}{n}}_1 + E_2\hat{\underset{\sim}{n}}_2) = \alpha \underset{\sim}{E}$$

Hence the polarization is parallel to the field even if the field direction is not along a symmetry axis.

9.22

$$P_x = \alpha_{xx} E_x + \alpha_{xy} E_y$$
$$P_y = \alpha_{yx} E_x + \alpha_{yy} E_y$$

$$dW = \mathbf{E}\cdot d\mathbf{p} = E_x\, dp_x + E_y\, dp_y$$
$$= E_x\left(\alpha_{xx} dE_x + \alpha_{xy} dE_y\right) + E_y\left(\alpha_{yx} dE_x + \alpha_{yy} dE_y\right)$$

Starting from $\mathbf{E}=0$, change $\mathbf{E}$ to $\mathbf{E} = \mathbf{E}_2$:

First method

first step: with $E_y = 0$, change E_x from 0 to E_{2x}

$E_y = 0 \quad dE_y = 0 \quad dW = E_x \alpha_{xx} dE_x$

$$\text{Work done} = \int dW = \alpha_{xx}\int_0^{E_{2x}} E_x\, dE_x = \tfrac{1}{2}\alpha_{xx} E_{2x}^2$$

second step: with $E_x = E_{2x}$, change E_y from 0 to E_{2y}:

$E_x = E_{2x} \quad dE_x = 0 \quad dW = E_{2x}\alpha_{xy} dE_y + E_y \alpha_{yy} dE_y$

$$\text{Work done} = E_{2x}\alpha_{xy}\int_0^{E_{2y}} dE_y + \alpha_{yy}\int_0^{E_{2y}} E_y\, dE_y = \alpha_{xy}E_{2x}E_{2y} + \tfrac{1}{2}\alpha_{yy}E_{2y}^2$$

Total work done $= \tfrac{1}{2}\alpha_{xx}E_{2x}^2 + \alpha_{xy}E_{2x}E_{2y} + \tfrac{1}{2}\alpha_{yy}E_{2y}^2$

Second method

first step: with $E_x = 0$, change E_y from 0 to E_{2y}

second step: with $E_y = E_{2y}$, change E_x from 0 to E_{2x}

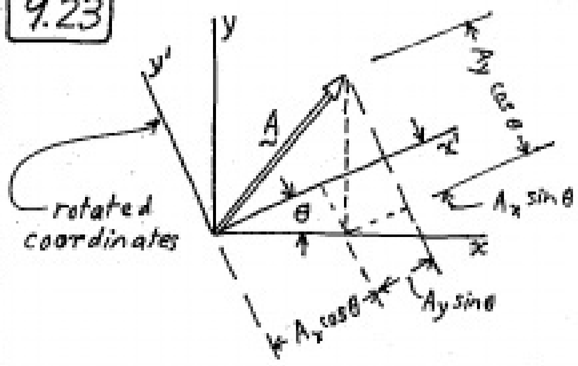
Obviously, we have merely interchanged the roles of x and y and the result is bound to be, if computed as above:

Total work done $= \tfrac{1}{2}\alpha_{xx}E_{2x}^2 + \alpha_{yx}E_{2y}E_{2x} + \tfrac{1}{2}\alpha_{yy}E_{2y}^2$

The work will be the same if and only if

$$\alpha_{xy} = \alpha_{yx}$$

9.23



Transformation of components:

① $A_{x'} = A_x\cos\theta + A_y\sin\theta$
$\quad A_{y'} = -A_x\sin\theta + A_y\cos\theta$

Likewise for any other vector $\mathbf{B}$:

② $B_{x'} = B_x\cos\theta + B_y\sin\theta$
$\quad B_{y'} = -B_x\sin\theta + B_y\cos\theta$

Let $\mathbf{B}$ and $\mathbf{A}$ be related linearly as follows:

$$B_x = \alpha_{xx}A_x + \alpha_{xy}A_y \quad ③$$
$$B_y = \alpha_{yx}A_x + \alpha_{yy}A_y \quad ④$$

the α's being constants. In rotated coordinates, the same relation between $\mathbf{A}$ and $\mathbf{B}$ will be expressed by

$$B_{x'} = \alpha_{x'x'}A_{x'} + \alpha_{x'y'}A_{y'} \quad ⑤$$
$$B_{y'} = \alpha_{y'x'}A_{x'} + \alpha_{y'y'}A_{y'} \quad ⑥$$

How must the constants $\alpha_{x'x'}$, etc. be related to α_{xx}, etc.? To discover this, use the component transformation ①, ② and the relations ③ and ④ to express all the vector components in ⑤ and ⑥ in terms of A_x and A_y. ⑤ then becomes:

$$(\alpha_{xx}A_x + \alpha_{xy}A_y)\cos\theta + (\alpha_{yx}A_x + \alpha_{yy}A_y)\sin\theta = \alpha_{x'x'}(A_x\cos\theta + A_y\sin\theta) + \alpha_{x'y'}(-A_x\sin\theta + A_y\cos\theta)$$

This must hold for any A_x, A_y. Set $A_y = 0$. Then

$$\alpha_{xx}\cos\theta + \alpha_{yx}\sin\theta = \alpha_{x'x'}\cos\theta - \alpha_{x'y'}\sin\theta \quad ⑦$$

Putting $A_x = 0$ gives another equation:

$$\alpha_{xy}\cos\theta + \alpha_{yy}\sin\theta = \alpha_{x'x'}\sin\theta + \alpha_{x'y'}\cos\theta \quad ⑧$$

Solve the simultaneous equations ⑦ and ⑧ for $\alpha_{x'x'}$ and $\alpha_{x'y'}$

⑨
$$\alpha_{x'x'} = \alpha_{xx}\cos^2\theta + \alpha_{yy}\sin^2\theta + (\alpha_{xy} + \alpha_{yx})\sin\theta\cos\theta$$
$$\alpha_{x'y'} = -\alpha_{xx}\sin\theta\cos\theta - \alpha_{yx}\sin^2\theta + \alpha_{xy}\cos^2\theta + \alpha_{yy}\sin\theta\cos\theta$$

Eq. (6) treated in the same way becomes:

$$-(\alpha_{xx}A_x+\alpha_{xy}A_y)\sin\theta + (\alpha_{yx}A_x+\alpha_{yy}A_y)\cos\theta = \alpha_{y'x'}(A_x\cos\theta + A_y\sin\theta)+\alpha_{y'y'}(-A_x\sin\theta + A_y\cos\theta)$$

Putting $A_y = 0$ gives the equation

$$-\alpha_{xx}\sin\theta+\alpha_{yx}\cos\theta = \alpha_{y'x'}\cos\theta - \alpha_{y'y'}\sin\theta \qquad (10)$$

while putting $A_x = 0$ gives the equation

$$-\alpha_{xy}\sin\theta + \alpha_{yy}\cos\theta = \alpha_{y'x'}\sin\theta+\alpha_{y'y'}\cos\theta \qquad (11)$$

Solving (10) and (11) for $\alpha_{y'x'}$ and $\alpha_{y'y'}$ gives:

$$(12)\quad \begin{cases} \alpha_{y'x'} = -\alpha_{xx}\sin\theta\cos\theta + \alpha_{yx}\cos^2\theta - \alpha_{xy}\sin^2\theta + \alpha_{yy}\sin\theta\cos\theta \\ \alpha_{y'y'} = \alpha_{xx}\sin^2\theta + \alpha_{yy}\cos^2\theta - \alpha_{yx}\sin\theta\cos\theta - \alpha_{xy}\sin\theta\cos\theta \end{cases}$$

Eqs. (9) and (12) give the complete prescription for the transformation of the components of the tensor α. Collecting them together:

$$\begin{cases} \alpha_{x'x'} = \alpha_{xx}\cos^2\theta + \alpha_{yy}\sin^2\theta + (\alpha_{xy}+\alpha_{yx})\sin\theta\cos\theta \\ \alpha_{y'y'} = \alpha_{xx}\sin^2\theta + \alpha_{yy}\cos^2\theta - (\alpha_{xy}+\alpha_{yx})\sin\theta\cos\theta \\ \alpha_{x'y'} = (-\alpha_{xx}+\alpha_{yy})\sin\theta\cos\theta + \alpha_{xy}\cos^2\theta - \alpha_{yx}\sin^2\theta \\ \alpha_{y'x'} = (-\alpha_{xx}+\alpha_{yy})\sin\theta\cos\theta - \alpha_{xy}\sin^2\theta + \alpha_{yx}\cos^2\theta \end{cases}$$

If $\alpha_{xy} = \alpha_{yx}$ the last two equations show that

$$\alpha_{x'y'} = \alpha_{y'x'} = (-\alpha_{xx}+\alpha_{yy})\sin\theta\cos\theta + \alpha_{xy}(\cos^2\theta - \sin^2\theta)$$

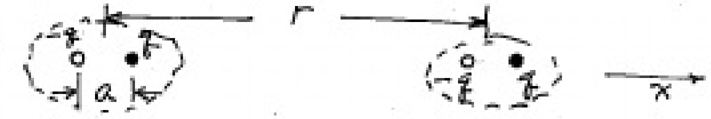
An angle θ can always be found which makes this zero: for

$$0 = (-\alpha_{xx}+\alpha_{yy})\sin\theta\cos\theta + \alpha_{xy}(\cos^2\theta - \sin^2\theta)$$

$$\frac{\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} = \frac{\alpha_{xy}}{\alpha_{xx}-\alpha_{yy}} \quad \text{or} \quad \tan 2\theta = \frac{2\alpha_{xy}}{\alpha_{xx}-\alpha_{yy}}$$

In fact, since $\tan 2(\theta+\frac{\pi}{2}) = \tan 2\theta$ there are four directions, $90°$ apart, for which $\alpha_{x'y'} = 0$.

9.24 (a) A pair of dipoles arranged like this $\vec{p} \xleftrightarrow{r} \vec{p}$ can be thought of as two pairs of point charges:



$p = qa \qquad r \gg a$

The net force on the pair of charges at the right is found from Coulomb's law:

$$F_x = q^2\left[\frac{2}{r^2} - \frac{1}{(r-a)^2} - \frac{1}{(r+a)^2}\right] = \frac{-q^2}{r^2}\left[\frac{1}{(1+\frac{a}{r})^2} + \frac{1}{(1-\frac{a}{r})^2} - 2\right]$$

Expanding $(1+\frac{a}{r})^{-2}$ and $(1-\frac{a}{r})^{-2}$ by the binomial theorem:

$$F_x = \frac{-q^2}{r^2}\left[1 - \frac{2a}{r} + \frac{3a^2}{r^2} + \dots + 1 + \frac{2a}{r} + \frac{3a^2}{r^2} + \dots - 2\right]$$

$$= -\frac{6q^2a^2}{r^4} + \text{higher powers}$$

In the limit, $a \to 0$ and $qa = p$, the dipole moment. Hence $F = \frac{-6p^2}{r^4}$; the minus sign indicates force to the left, or <u>attraction</u>.

The same result is obtained more quickly by applying Eq. 22, p. 308, for the force on a dipole $\mathbf{p}$ in a field $\mathbf{E}$:

$$F_x = \mathbf{p}\cdot \text{grad}\, E_x\ ;\ F_y = \mathbf{p}\cdot\text{grad}\, E_y\ ;\ F_z = \mathbf{p}\cdot\text{grad}\, E_z$$

Here $r = x$ $\quad E_x = \frac{2p}{x^3}$ (field of a dipole on its axis) and we have: $F_x = p\frac{\partial}{\partial x}\left(\frac{2p}{x^3}\right) = \frac{-6p^2}{x^4}$.

(b) Dipoles oriented like this $\uparrow \xleftrightarrow{r} \uparrow$ can so be treated as pairs of close point charges. is obvious that the net force is repulsive, for distance between unlike charges is greater than

the distance between like charges and is also at a slight angle with respect to the x-direction.



The net force on the right-hand pair is:

$$F_x = \frac{2q^2}{x^2} - \frac{2q^2}{(x^2+a^2)}\frac{x}{(x^2+a^2)^{1/2}} = \frac{2q^2}{x^2}\left[1 - \frac{1}{\left(1+\frac{a^2}{x^2}\right)^{3/2}}\right]$$

$$= \frac{2q^2}{x^2}\left[1-\left(1-\frac{3}{2}\frac{a^2}{x^2}+\cdots\right)\right] = \frac{3q^2a^2}{x^4} = \frac{3p^2}{x^4}.$$

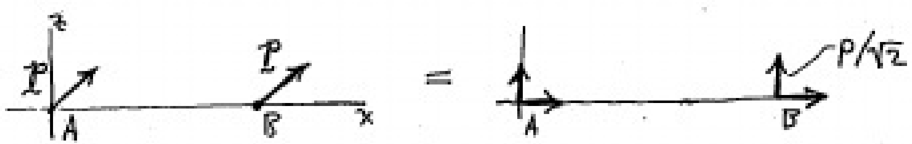
This also can be computed from the dipole field directly. In this arrangement, the field of the dipole on the left has, as its x-component in the vicinity of the dipole on the right,

$$E_x = \frac{3pxz}{(x^2+z^2)^{5/2}}$$ (see Eq. 14, p. 306)

$F_x = \mathbf{p}\cdot \text{grad}\, E_x = p\frac{\partial E_x}{\partial z}$ (since $\mathbf{p}$ is in the z direction) $\frac{\partial E_x}{\partial z} = \frac{3px}{(x^2+z^2)^{5/2}} - \frac{15pxz^2}{(x^2+z^2)^{7/2}} = \frac{3p}{x^4}$ at $z=0$

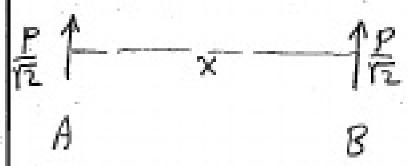
Hence $F_x = \frac{3p^2}{x^4}$ in agreement with the result above.

(c) For dipoles arranged like this ↗------↗, one way to compute the force is to consider each dipole as composed of two dipoles:



The force on B is then the sum of four forces, each of which is the force on the dipole on the

right in one of the arrangements below:



From result in (b)
$F_x = 3\left(\frac{p}{\sqrt2}\right)^2\frac{1}{x^4}$ $F_z = 0$

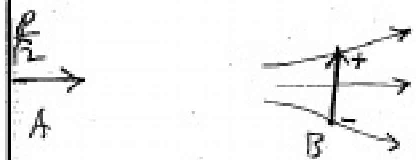


From result in (a)
$F_x = -6\left(\frac{p}{\sqrt2}\right)^2\frac{1}{x^4}$ $F_z = 0$



This is new. Force is evidently up, because E_z is negative and is stronger at the − end. $E_z = -\left(\frac{p}{\sqrt2}\right)\frac{1}{x^3}$

So $F_z = 3\left(\frac{p}{\sqrt2}\right)\frac{1}{x^4}\left(\frac{p}{\sqrt2}\right)$, $F_x = 0$



Force is up here also, and must have same magnitude as in preceding arrangement, because A and B have merely been interchanged.

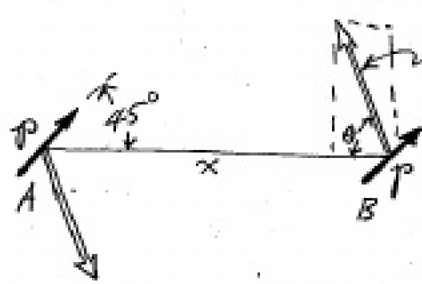
$F_z = 3\left(\frac{p}{\sqrt2}\right)^2\frac{1}{x^4}$ $F_x = 0$

The resultant force is:

$$F_x = 3\left(\frac{p}{\sqrt2}\right)^2\frac{1}{x^4} - 6\left(\frac{p}{\sqrt2}\right)^2\frac{1}{x^4} = -\frac{3}{2}\frac{p^2}{x^4}$$

$$F_z = 3\left(\frac{p}{\sqrt2}\right)^2\frac{1}{x^4} + 3\left(\frac{p}{\sqrt2}\right)^2\frac{1}{x^4} = 3\frac{p^2}{x^4}$$

The force on dipole A is equal and opposite:



$F_x = -\frac{3}{2}\frac{p^2}{x^4}$

$F_z = +3\frac{p^2}{x^4}$

$\theta = \tan^{-1} 2 = 63.5°$

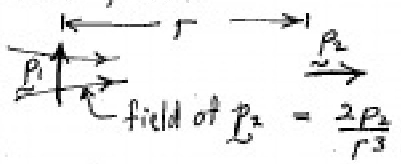
9.25



(b, p. 439)

No work is required to bring in p_2 in this orientation, for the force on it is perpendicular to the path.

(c)



The field of p_2 at the location of p_1 is $\frac{2p_2}{r^3}$, horizontal, toward the right.

(d)

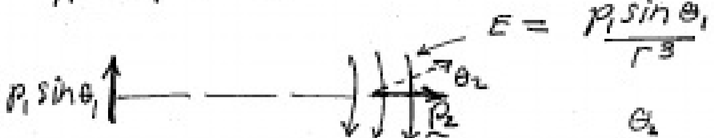


The work required to rotate p_1 from its orientation in (c) to its orientation in (d) is negative (p_1 "wants" to point in the direction of the field) Torque = $\frac{2p_2}{r^3}\cdot p_1 \cdot \sin\theta$

$$\text{Work} = \int_{\frac{\pi}{2}}^{\theta_1} \frac{2p_2 p_1}{r^3}\sin\theta\, d\theta = -\frac{2p_2 p_1}{r^3}\cos\theta_1$$

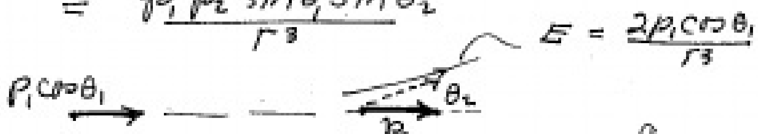
Now, with p_1 fixed, resolve it into components and consider these separately in calculating the work required to turn p_2 into its final position:





$$\text{Work to rotate } p_2 \text{ up to } \theta_2 = \int_0^{\theta_2} p_2 E\cos\theta\, d\theta$$

$$= \frac{p_1 p_2 \sin\theta_1 \sin\theta_2}{r^3}$$



$$\text{Work to rotate } p_2 \text{ up to } \theta_2 = \int_0^{\theta_2} p_2 E \sin\theta\, d\theta$$

$$= \frac{2p_1 p_2}{r^3}\cos\theta_1(1-\cos\theta_2)$$

Adding up all increments of work done:

$$W = -\frac{2p_1p_2\cos\theta_1}{r^3} + \frac{p_1p_2\sin\theta_1\sin\theta_2}{r^3} + \frac{2p_1p_2}{r^3}\cos\theta_1(1-\cos\theta_2)$$

$$= \frac{p_1p_2}{r^3}\left[-2\cos\theta_1 + \sin\theta_1\sin\theta_2 + 2\cos\theta_1 - 2\cos\theta_1\cos\theta_2\right]$$

$$= \frac{p_1p_2}{r^3}\left[\sin\theta_1\sin\theta_2 - 2\cos\theta_1\cos\theta_2\right]$$

$$= \frac{p_1p_2}{r^3}\left[\cos(\theta_1-\theta_2) - 3\cos\theta_1\cos\theta_2\right]$$

9.26 Applying the result of Prob. 9.25, for a pair of dipoles like this:



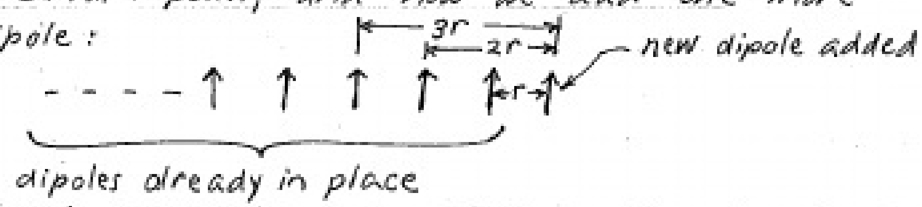
$\theta_1 = \theta_2 = \frac{\pi}{2}$ $\quad W = +\frac{p_1p_2}{r^3}$

For dipoles like this:



$\theta_1 = \theta_2 = 0$ and $W = -\frac{2p_1p_2}{r^3}$.

The array ↑ ↑ ↑ ↑ ↑ . . . . (a) has positive energy, clearly. To calculate the energy per dipole, which we shall denote by $\overline{W}$, suppose the array has been assembled from the left, up to a certain point, and now we add one more dipole:



This increases the energy of the system by an amount: $\frac{p^2}{r^3} + \frac{p^2}{(2r)^3} + \frac{p^2}{(3r)^3} + \cdots$ But so does every subsequent dipole added. Hence the

energy of this array, per dipole, is:

$$\overline{W}_{(a)} = \frac{p^2}{r^3}\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\cdots\right)$$

The other arrays can be analyzed in the same way:

(b) . . . . ↑ ↓ ↑ ↓ ↑

$$\overline{W}_{(b)} = \frac{p^2}{r^3}\left(-1+\frac{1}{2^3}-\frac{1}{3^3}+\frac{1}{4^3}\cdots\right)$$

(c) . . . . → → → →

$$\overline{W}_{(c)} = -\frac{2p^2}{r^3}\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\cdots\right)$$

(d) . . . . → ← → ← →

$$\overline{W}_{(d)} = \frac{2p^2}{r^3}\left(1-\frac{1}{2^3}+\frac{1}{3^3}-\frac{1}{4^3}+\cdots\right)$$

Obviously (c) has the lowest energy. To decide whether (a) or (d) is the array with the highest energy we have to compare the series

$$2\left(1-\frac{1}{8}+\frac{1}{27}-\frac{1}{64}\cdots\right) \quad \text{with} \quad \left(1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}\cdots\right)$$

A little arithmetic convinces one that $2(1-\frac{1}{8}+\frac{1}{27}\cdots)$ sums to a larger number. [If a rigorous proof were required, here is one: consider the integral

$$\int_{n-1}^{n}\frac{dx}{x^3} = \frac{n-\frac{1}{2}}{n^2(n-1)^2}$$

which is greater than $\frac{1}{n^3}$.

In the series $1+\frac{1}{8}+\frac{1}{27}+\cdots$ replace every term $\frac{1}{n^3}$ with the larger quantity $\int_{n-1}^{n}\frac{dx}{x^3}$, beginning with the term $\frac{1}{27}$. Then:

$$1+\frac{1}{8}+\frac{1}{27}+\cdots < 1+\frac{1}{8}+\int_{2}^{\infty}\frac{dx}{x^3} = 1+\frac{1}{8}+\frac{1}{8} = \frac{5}{4}$$

But the alternating series $2(1-\frac{1}{8}+\frac{1}{27}-\cdots)$ $> 2(1-\frac{1}{8}) = \frac{7}{4}$.]

Thus (d) has the highest energy.

9.27 Let us start with all dipoles pointing the same way:

|← r →|

· · · → → → → → → → → → · · ·
(the sixth arrow from the left is labelled p; the arrows from the fifth on are numbered 1, 2, 3, 4, 5)

We are going to turn the numbered dipoles end-for-end, starting with 1. The field at number 1, due to all the other dipoles is $E = 2p\left(\frac{2}{r^3}+\frac{2}{(2r)^3}+\frac{2}{(3r)^3}+\cdots\right)$

The work required to reverse dipole 1 in this field is $2pE$ or

$$\frac{8p^2}{r^3}\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\cdots\right) = W_1$$

The array now looks like this:

· · · → → → → ← → → → → · · ·
(the reversed arrow ← is no. 1; the arrows after it are numbered 2, 3, 4, 5)

At number 2, the fields of no. 1 and no. 3 cancel each other. Hence the field at 2 is now $2p\left(\frac{2}{(2r)^3}+\frac{2}{(3r)^3}+\cdots\right)$ and the work done in turning 2 around is:

$$\frac{8p^2}{r^3}\left(\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\cdots\right) = W_2$$

Similarly, $W_3 = \frac{8p^2}{r^3}\left(\frac{1}{3^3}+\frac{1}{4^3}+\cdots\right)$

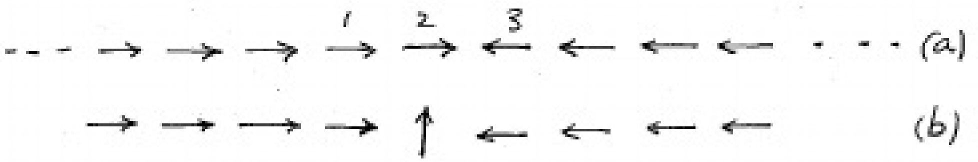
Thus: $W_1+W_2+W_3+\cdots = \frac{8p^2}{r^3}\left[1+\frac{2}{2^3}+\frac{3}{3^3}+\frac{4}{4^3}+\cdots\right]$

$$= \frac{8p^2}{r^3}\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right] = \frac{8p^2}{r^3}\cdot\frac{\pi^2}{6}$$

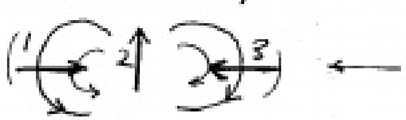
Hence the energy associated with the "boundary," which is the difference between the energy of an array with such a "boundary" and one without, is $\Delta W = \frac{4\pi^2}{3}\frac{p^2}{r^3}$.

However, this is also the energy of two boundaries when they are far apart, for in creating one boundary

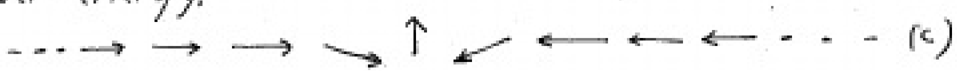
we automatically create two. That is, if we stop the turning-over routine described above after the n^{th} dipole has been turned over, we have two boundaries separated by a distance nr, and the work done is $W_1 + W_2 + \cdots + W_n$, which is somewhat less than $\frac{4\pi^2}{3}\frac{\mu^2}{r^3}$. The energy associated with a pair of boundaries as close together as possible is just W_1, the work required to turn over one dipole in the original uninterrupted chain. Evidently the energy associated with a pair of boundaries is lower when they are close together than when they are far apart, which implies that they will tend to attract one another. (Such boundary pairs are rather like electron-positron, or electron-"hole" pairs.)



Since the field at dipole 2 is zero in (a), no work is done in turning this dipole through 90°, and (b) has therefore the same energy as (a). But now consider the direction of the field at dipole 3, due to the other dipoles including 2.



The field of 2 has a downward component at both 1 and 3. This makes these want to tip down. That is, the configuration (c) is now one of slightly lower energy.



9.28 From Eq. 72: $P = \dfrac{N\alpha}{1 - \frac{4\pi}{3}N\alpha}E$

$$\epsilon = 1 + 4\pi\frac{P}{E}$$

$$\epsilon = 1 + \frac{4\pi N\alpha}{1 - \frac{4\pi}{3}N\alpha}$$

$\epsilon - 1 = \dfrac{x}{1 - \frac{x}{3}}$ where $x = 4\pi N\alpha$. Solve for x:

$$x = \frac{3(\epsilon - 1)}{\epsilon + 2} \quad \text{or} \quad 4\pi\alpha = \frac{3}{N}\left(\frac{\epsilon - 1}{\epsilon + 2}\right)$$

To test the formula, we shall see whether the quantity $\frac{1}{N}\left(\frac{\epsilon - 1}{\epsilon + 2}\right)$ is a constant. N is directly proportional to the density d, so we can just as well look at $\frac{1}{d}\frac{(\epsilon - 1)}{(\epsilon + 2)}$. From the data given:

density	$\epsilon - 1$	$\epsilon + 2$	$(\epsilon-1)/(\epsilon+2)d$
.00118	.00052	3.00052	0.147
.06604	.03109	3.03109	0.1553
.2361	.11413	3.11413	0.1552
.5780	.29633	3.29633	0.1555

The agreement is pretty good, the lack of constancy in the last column being only slightly greater than the uncertainty in the data implied by the number of significant figures given.

9.29



polarizability α

Suppose A has a dipole moment p_A in the direction shown. It will produce a field at B given by $E = \frac{2p_A}{r^3}$. This will induce in B a dipole moment $p_B = \alpha E = \frac{2\alpha p_A}{r^3}$, which in turn will cause an electric field at A of magnitude $\frac{2p_B}{r^3}$,

in the direction favorable for maintaining the moment p_A. E at $A = \frac{2p_B}{r^3} = \frac{4\alpha p_A}{r^6}$

This would induce a moment

$$p_A = \alpha E = \frac{4\alpha^2 p_A}{r^6}$$

induced ↑ (left side) ← assumed (right side)

Thus the critical value of α would appear to be: $\frac{4\alpha^2}{r^6} = 1$ or $\alpha = \frac{r^3}{2}$. But we know that α is of order R^3 where R is the atomic radius. Hence the implied condition for "amplification" can only be satisfied for a distance r so small as to be of the order of atomic size. At that distance the atom cannot be treated as a point dipole.

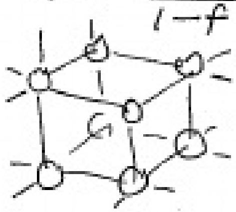
If the equations were valid, with $4\alpha^2/r^6 > 1$, they would imply infinite polarization, the energy going more and more negative without limit.

9.30 For the conducting sphere of radius a, $\alpha = a^3$ (see Problem 9.9). Assume we have N spheres per unit volume. Then $\frac{4\pi}{3}a^3 N$ is the volume of the spheres contained in unit volume, or the fraction of volume, f, occupied.

$$\epsilon = 1 + \frac{4\pi N\alpha}{1 - \frac{4\pi}{3}N\alpha} = 1 + \frac{4\pi N a^3}{1 - \frac{4\pi}{3}Na^3} = 1 + \frac{3f}{1-f}$$

$$\epsilon = \frac{1+2f}{1-f}$$

Since f is necessarily <1, the fraction can't blow up.



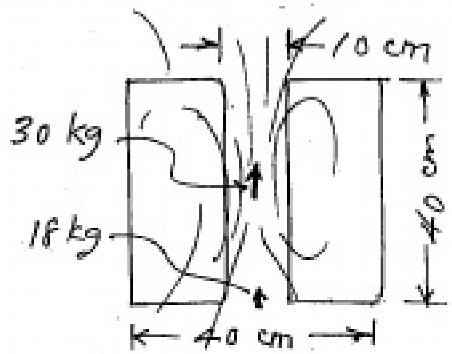
For 1 in. diameter spheres on 2 in. centers,

$$f = \frac{\text{volume of 1" sphere}}{\text{volume of 2" cube}} = \frac{\pi}{6}/8 = 0.065$$

$$\epsilon = 1 + \frac{3(.065)}{1-.065} = 1.209$$

CHAPTER 10

10.1 The field strength on the axis is 30 kilogauss at the center of the coil, 18 kilogauss at the end, according to the description on page 352. The energy stored is given exactly by $\frac{1}{8\pi}\int B^2 dv$ over all space,



including hollow core, windings, and space outside. As a very rough estimate, we might take $B = 18000$ over the whole cylinder occupied by the coil and zero elsewhere, giving

$$U \approx \frac{1}{8\pi}(18000)^2(\pi \times 20^2 \times 40) = 6.5 \times 10^{11} \text{ ergs}$$

Rounding it off: $U \approx 6 \times 10^4$ joules *

If power is supplied at the rate of 400 kilowatts, the time required to build up the field must be, in order of magnitude, $\frac{6 \times 10^4 \text{ joules}}{4 \times 10^5 \text{ watts}} = 0.15$ sec.

* The true value is 4.8×10^4 joules. This can be found by using the formulas and tables for the inductance of multiple layer solenoids in the "Handbook of Chemistry and Physics."

10.2 $B_z = 18000$ gauss $\quad \frac{dB_z}{dz} = 1700$ gauss/cm

At 90°K the force on 1 gm of O_2 was 7.5×10^3 dynes.

$F = m\frac{dB}{dz} \quad m = \chi B \quad F = \chi B \frac{dB}{dz}$, from which

$$\chi = \frac{7.5 \times 10^3}{18 \times 10^3 \times 1.7 \times 10^3} = 2.45 \times 10^{-4} \quad \text{(susceptibility per gm)}$$

1 cm³ of air at NTP contains $\left(\frac{1}{5}\right)\left(\frac{32}{22.4 \times 10^3}\right)$ gms O_2

or 2.85×10^{-4} gms. The ratio of the absolute temperatures is $\frac{273}{90}$. Thus the volume susceptibility of air at NTP should be

$$2.45 \times 10^{-4} \left(\frac{90}{273}\right)\left(2.85 \times 10^{-4}\right) = 2.30 \times 10^{-8}$$

10.3 Eq. 6.41, p. 202: $B_z = \dfrac{2\pi b^2 I}{c(b^2+z^2)^{3/2}}$

For $z \gg b$, we can neglect b^2 compared to z^2 in the denominator, and the formula reduces to

$$B_z \approx \frac{2\pi b^2 I}{c z^3}$$

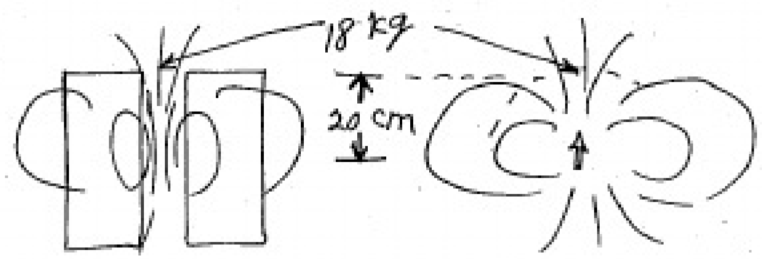
Since πb^2 is the area of the loop, its magnetic moment m is $\frac{\pi b^2 I}{c}$, and the expression above can be written $B_z \approx \frac{2m}{z^3}$, which is just the field of a dipole on its axis.

More exactly, the factor $(b^2+z^2)^{-3/2}$ can be expanded in powers of $\left(\frac{b}{z}\right)^2$:

$$B = \frac{2\pi b^2 I}{c z^3}\left(1+\frac{b^2}{z^2}\right)^{-3/2} = \frac{2\pi b^2 I}{c z^3}\left(1 - \frac{3}{2}\frac{b^2}{z^2} + \cdots\right)$$

from which we see that the field will be within 1% of the dipole field if $\frac{3}{2}\frac{b^2}{z^2} < \frac{1}{100}$, or $z > \sqrt{150\, b^2} \approx 12\, b$.

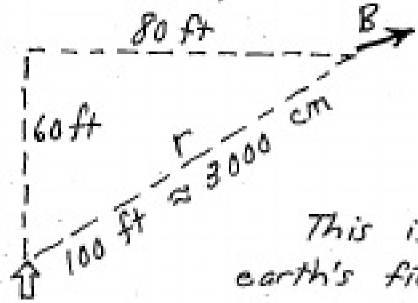
10.4



To estimate roughly the magnetic dipole moment

of the solenoid, let us suppose that it is equivalent to a point dipole which would produce, 20 cm away on its axis, a field strength B_z equal to that at the end of the solenoid, namely 18000 gauss. This is reasonable because the magnetic field configuration near the end of the solenoid and beyond looks not very different from a dipole field. On this assumption,

$$18000 = \frac{2m}{(20)^3}, \quad \text{or} \quad m = 7.2 \times 10^{7} \text{ cgs units *}$$



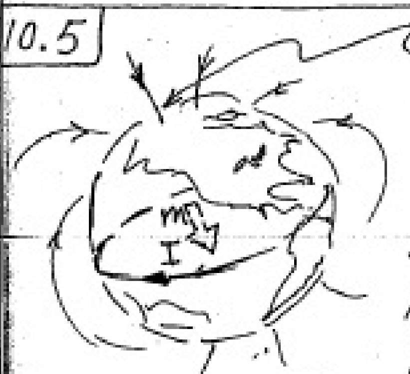
In order of magnitude,

$$B \approx \frac{m}{r^3} = \frac{7\times10^7}{27\times10^9} = 2.5\times10^{-3} \text{ gauss}$$

This is small compared to the earth's field, and if it were perfectly steady could not be noticed. But if frequently switched on and off it might cause trouble.

* An exact calculation, taking into account the actual current distribution in the coil, gives for its dipole moment the value 6.2×10^7, so the above estimate is pretty good.

10.5



$B = 0.62$ gauss

$r = 6400$ km $= 6.4 \times 10^8$ cm

For central dipole of strength m, $B = \frac{2m}{r^3}$

Hence $m = \frac{Br^3}{2}$

$$m = \frac{(0.62)(6.4\times10^8)^3}{2} = 8\times10^{25} \text{ cgs units}$$

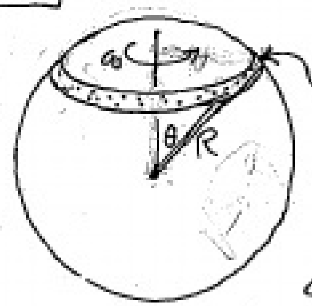
Current I amps flowing in ring of radius r has dipole moment $m = \frac{\pi I r^2}{10}$. Hence the equatorial current required to make a dipole moment

as large as 8×10^{25} would be:

$$I = \frac{10}{\pi}\frac{m}{r^2} = \frac{10}{\pi}\frac{(8\times10^{25})}{(6.4\times10^8)^2} = 6\times10^8 \text{ amperes.}$$

[Of course, if a concentrated equatorial current were the source of the field, the field would be somewhat different from that of a central dipole, and the polar field would be $\frac{2\pi I r^2}{10(r^2+r^2)^{3/2}}$ or $\frac{\pi}{10} I r^2 \cdot \frac{1}{\sqrt{2}\, r^3}$]

10.6



Total charge Q distributed on surface with density $\sigma = Q/4\pi R^2$.

area of band $= 2\pi R^2 \sin\theta\, d\theta$

charge in band $= dQ$

$$dQ = \sigma \cdot 2\pi R^2 \sin\theta\, d\theta = \frac{Q}{2}\sin\theta\, d\theta$$

Current = charge passing a given point, per second. If the sphere rotates at angular velocity ω radians/sec, the band is equivalent to a current dI, in esu/sec, of magnitude:

$$dI = \left(\frac{\omega}{2\pi}\right) dQ = \frac{\omega Q}{4\pi}\sin\theta\, d\theta$$

The magnetic moment of this current ring is

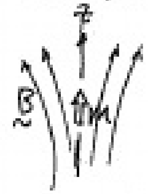
$$dm = \frac{dI}{c}\pi(R\sin\theta)^2 = \frac{\omega Q R^2}{4c}\sin^3\theta\, d\theta$$

Integrating over the sphere to get the total magnetic moment:

$$m = \frac{\omega Q R^2}{4c}\int_0^{\pi}\sin^3\theta\, d\theta = \frac{\omega Q R^2}{2c}\int_0^{\pi/2}(1-\cos^2\theta)\sin\theta\, d\theta$$

$$= \frac{\omega Q R^2}{2c}\int_0^1 (1-x^2)\,dx = \frac{\omega Q R^2}{2c}\cdot\frac{2}{3} = \frac{\omega Q R^2}{3c}$$

10.8



Magnetic force on dipole of moment m is

$$F = m\frac{dB}{dz}$$ if m is on the axis, parallel to B.

Also, $m = \chi B$ for 1 gm of material, if χ is the specific susceptibility.

Then $F = \chi B \frac{dB}{dz}$. The work done against this force is

$$W = -\int F\,dz = -\chi\int B\frac{dB}{dz}\,dz = -\frac{\chi}{2}\int d(B^2)$$

Thus if one gram of material is moved along the axis from a point where the field is B_1 to a point where the field is B_2, the work done is $\frac{\chi}{2}(B_1^2 - B_2^2)$.

If B_2^2 can be neglected in comparison with B_1^2, we have $W = \frac{\chi}{2}B_1^2$.

This can also be written in terms of the force F_1 and the gradient $\left(\frac{dB}{dz}\right)_1$ at the initial position:

$$\chi B_1 = \frac{F_1}{\left(\frac{dB}{dz}\right)_1}, \quad \text{or} \quad W = \frac{1}{2}\frac{F_1 B_1}{\left(\frac{dB}{dz}\right)_1}$$

Substituting the data given in the table for liquid oxygen: $F_1 = 7.5\times10^3$ dynes, for $B_1 = 18\times10^3$ gauss and $\left(\frac{dB}{dz}\right)_1 = 1700$ gauss/cm, we have:

$$W = \frac{1}{2}\,\frac{7.5\times10^3\times18\times10^3}{1700} = 4.0\times10^4 \text{ ergs}$$

10.9 $F = m\frac{dB}{dz}$ $\quad m = \frac{F}{\left(\frac{dB}{dz}\right)}$ $\quad F = 4\times10^5$ dynes, $\left(\frac{dB}{dz}\right) = 1700\ \frac{\text{gauss}}{\text{cm}}$

$m = \frac{4\times10^5}{1700} = 235$ cgs units (ergs/gauss)

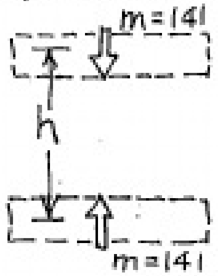
no of atoms per gm $= \frac{6\times10^{23}}{56} = 1.07\times10^{22}$

electron spin moment $= 0.93\times10^{-20}$ cgs units.

effective no. of spins per atom $= \frac{235}{1.07\times10^{22}\times0.93\times10^{-20}} = 2.35$

10.10 Volume of disk $= \pi \times (1\text{ cm})^2 \times 0.3\text{ cm} = 0.94\text{ cm}^3$

Magnetic moment: $m = MV = 150 \times 0.94 = 141$ ergs/gauss



Field of lower dipole at position of upper dipole: $B = \dfrac{2m}{h^3}$

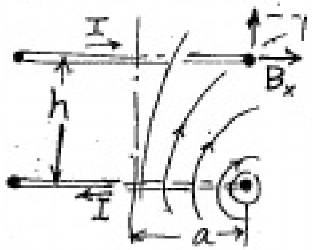
$$\text{Force} = m\frac{dB}{dh} = \frac{6m^2}{h^4} = \frac{6\times(141)^2}{h^4}\text{ dynes}$$

If we assume the disk has the density of iron, 7.8 gm/cm³, the weight of the disk in dynes is $7.8 \times 0.94 \times 980 = 7000$ dynes. Then the height at which the upper disk will "float" is determined by

$$\frac{6\times(141)^2}{h^4} = 7000 \quad\text{or}\quad h = \left(\frac{120\,000}{7000}\right)^{\frac14} = 2.0\text{ cm}$$

(At that position the disk will be stable vertically, but unstable horizontally.)

Since the height thus predicted is not very large compared to the radius of the disks, the approximation made in treating the disks as concentrated dipoles is not very good. In fact, each disk is precisely equivalent to a band of current, as shown in Fig. 10.22 (c). To a very good approximation, we could treat the band as a ring of current, and we could find the force on the upper current ring if we could calculate the horizontal magnetic field component B_x of the field of the lower ring. This can be done, using the Biot-Savart formula, Eq. 38, p. 200, with the result that:



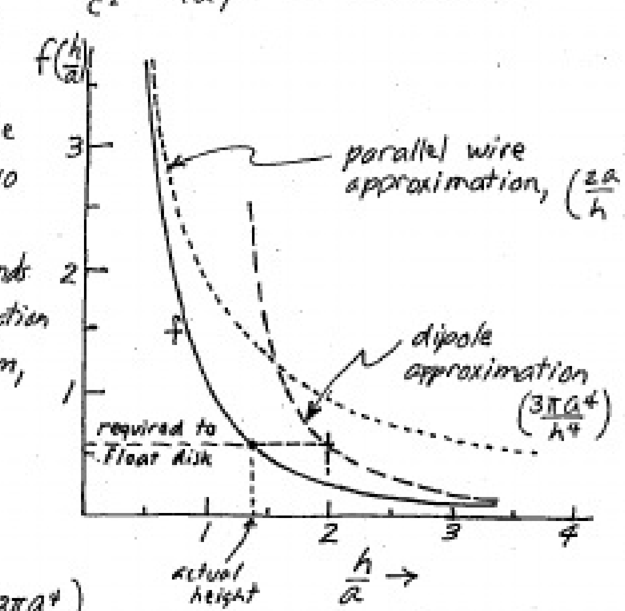
$$B_x = \frac{Ih}{2ca^2}\,k^3\int_0^{\pi/2}\frac{(\sin^2\theta - \cos^2\theta)\,d\theta}{(1-k^2\sin^2\theta)^{3/2}}$$

where $k^2 = \dfrac{4a^2}{4a^2+h^2}$. The integral can be expressed in terms of the complete elliptic integrals $E(k^2)$ and $K(k^2)$ defined by $E = \int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}\,d\theta$ and $K = \int_0^{\pi/2}\dfrac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$

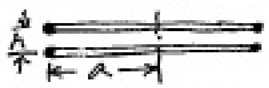
(see, for instance, Gröbner & Hofreiter, Unbestimmte Integrale, page 67),

and the final result for the force on the upper ring, which is $\dfrac{2\pi a I}{c}B_x$, is: $F = \dfrac{2\pi I^2}{c^2}\dfrac{h}{\sqrt{h^2+4a^2}}\left[\dfrac{2-k^2}{1-k^2}E - 2K\right]$

Let's write this as $F = \dfrac{2\pi I^2}{c^2} f\!\left(\dfrac{h}{a}\right)$. A graph of f as a function of $\dfrac{h}{a}$ is sketched at the right. On the same graph are shown two approximations. The dashed curve corresponds to the dipole approximation we used in the problem, according to which $F = \dfrac{6m^2}{h^4}$. Since $m^2 = \left(\dfrac{\pi a^2 I}{c}\right)^2$, this implies $F = \dfrac{2\pi I^2}{c^2}\cdot\left(\dfrac{3\pi a^4}{h^4}\right)$



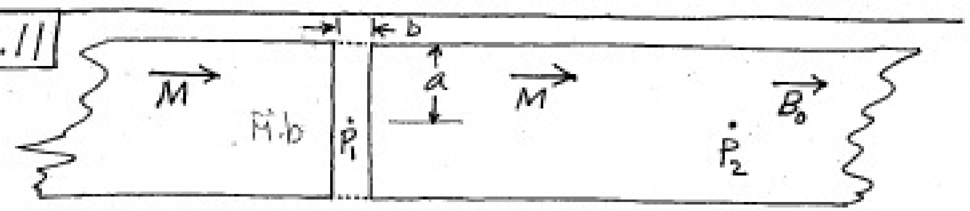
The dotted curve would be a good approximation if the current rings were very close together:



Then we could treat them as long parallel wires, for which $B_x = \dfrac{2I}{ch}$ and $F = \dfrac{2I}{ch}\cdot\dfrac{2\pi a I}{c} = \dfrac{2\pi I^2}{c^2}\cdot\left(\dfrac{2a}{h}\right)$

For the conditions given, we see that the correct height is $h = 1.4\,a$, rather than $h = 2.0\,a$, as given by our dipole approximation.

10.11



The equivalent current is a cylindrical current sheath, like a long solenoid, with a short section missing. The missing section has negligible effect at P_2, so the

$B_0 = \mu_0 M = \mu_0 m \frac{\pi R^2 L}{\pi R^2}$

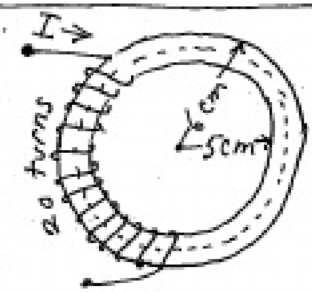
field there is simply: $B_0 = 4\pi M$. It would have the same value at P_1 if the missing current band were restored. The current density in the sheath is: $J = cM$, hence the missing current amounts to: $I = bJ = bcM$. Its field at P_1, the center of the current ring, would be:

$$\Delta B = \frac{2\pi I}{ca} = \frac{2\pi M b}{a}.$$

Hence the field at P_1 is actually:

$$B = 4\pi M - \frac{2\pi M b}{a} = B_0\left(1 - \frac{b}{2a}\right)$$

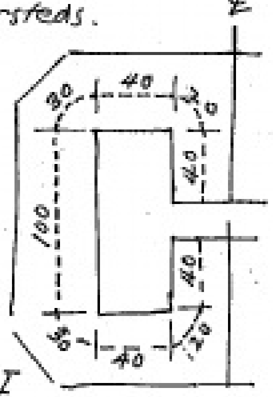
10.12 $B = 12000$ gauss



According to the B-H curve in Fig. 10-30, this requires $H = 1.6$ oersteds. The mean circumference is 11π cm. If I is in amperes, $\int H\,dl = \frac{4\pi NI}{10}$, or

$$11\pi \times 1.6 = \frac{4\pi}{10} \cdot 20 I \qquad I = \frac{10 \times 11\pi \times 1.6}{20 \times 4\pi} = 2.2 \text{ amp}$$

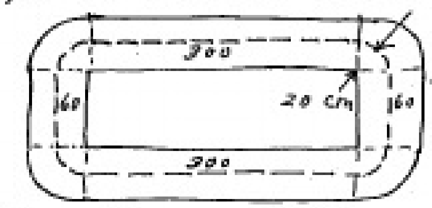
10.13 From the B-H curve on page 403 we find that $B = 16\,000$ gauss for $H = 63$ oersteds. From the dimensions given, we estimate a path length in iron of 360 cm. In the gap, of length 20 cm $H = B = 16\,000$.



$$\int H\,dl = \overbrace{360 \times 63}^{\text{iron}} + \overbrace{20 \times 16\,000}^{\text{gap}}$$
$$= 23{,}000 + 320{,}000$$
$$= 343{,}000 \text{ oersted-cm} = \frac{4\pi NI}{10}$$

$$NI = \frac{10 \cdot 343{,}000}{4\pi} = 270{,}000 \text{ ampere turns}$$

To determine the power required we first calculate the average length of a turn in the coil:



$$L = 300 + 300 + 60 + 60 + 4\left(\frac{\pi}{2}\right)20$$
$$= 845 \text{ cm}$$

Suppose we have N turns, including both upper and lower coil. Total cross-section area of copper is 1500 cm^2. Then the resistance is: $R = N \times \frac{845\rho}{\left(\frac{1500}{N}\right)} = N^2 \times \frac{845}{1500}\rho$

$$\text{Power} = I^2 R = I^2 N^2 \frac{845}{1500}\rho$$

$\rho = 2 \times 10^{-6}$ ohm-cm and $NI = 270\,000$ ampere turns

Hence $P = \frac{(2.7 \times 10^5)^2 \times 845 \times 2 \times 10^{-6}}{1500} = 82.5$ kilowatts

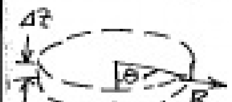
$P = IV$ with V given as 400 volts.

Then $I = P/V = \frac{82500}{400} = 206$ amperes

$N = \frac{NI}{I} = \frac{270000}{206} = 1310$ turns or 655 per coil

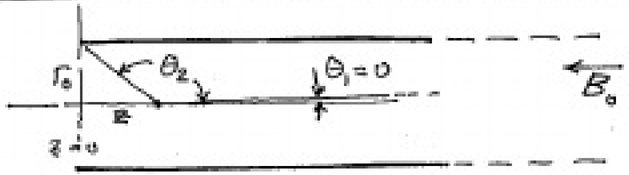
10.14 Eq. 18: $F = m \frac{\partial B_z}{\partial z}$





Consider the pill-box in Fig. 10.10 and let B_r vary as some function of θ, instead of being constant. The flux from the side of the box is then $\Delta z \int_0^{2\pi} B_r\, r\, d\theta$. But the force on the current ring depends in exactly the same way on the average around the circle. That is, since $dl = r\,d\theta$, $F = \int \frac{I}{c} B_r\, dl = \frac{I}{c}\int_0^{2\pi} B_r\, r\, d\theta$. Hence the result is not changed if B_r is not constant.

10.15



Eq. 6.44, p. 203: $B_z = \frac{2\pi I n}{c}(\cos\theta_1 - \cos\theta_2)$

Let B_0 be the field strength far back inside the solenoid, where $\theta_2 = \pi$, $\theta_1 = 0$. $B_0 = \frac{4\pi I n}{c}$

Near the end, $B = \frac{B_0}{2}(1 - \cos\theta_2)$ since $\theta_1 = 0$.

$\cos\theta_2 = \frac{-z}{\sqrt{z^2+r_0^2}}$, so that $B = \frac{B_0}{2}\left(1 + \frac{z}{\sqrt{z^2+r_0^2}}\right)$

For a sample of constant susceptibility χ, the force is proportional to $B\frac{dB}{dz}$, and will be maximum where $\frac{d}{dz}\left(B\frac{dB}{dz}\right) = 0$

$$\frac{d}{dz}B\frac{dB}{dz} = B\frac{d^2B}{dz^2} + \left(\frac{dB}{dz}\right)^2$$

$$B = \frac{B_0}{2}\left(1 + \frac{z}{\sqrt{z^2+r_0^2}}\right) \qquad \frac{dB}{dz} = \frac{B_0 r_0^2}{2(z^2+r_0^2)^{3/2}}$$

$$\frac{d^2B}{dz^2} = \frac{-3B_0 r_0^2 z}{2(z^2+r_0^2)^{5/2}}$$

Setting $\frac{d}{dz}\left(B\frac{dB}{dz}\right) = 0$ we have:

$$\frac{1}{2}\left(1 + \frac{z}{\sqrt{z^2+r_0^2}}\right)\left(\frac{-3r_0^2 z}{2(z^2+r_0^2)^{5/2}}\right) + \frac{r_0^4}{4(z^2+r_0^2)^3} = 0$$

Clearing: $r_0^2 = 3z\left(\sqrt{z^2+r_0^2} + z\right)$

$r_0^2 - 3z^2 = 3z\sqrt{z^2+r_0^2}$ squaring:

$r_0^4 - 6r_0^2 z^2 + 9z^4 = 9z^4 + 9z^2 r_0^2$

$r_0^2 = 15z^2 \qquad z = \pm\frac{r_0}{\sqrt{15}}$

Looking back at the equation we had before squaring, it is clear that only the positive root of the last equation is a solution: $z = +\frac{r_0}{\sqrt{15}}$. The point in question lies just inside the end of the solenoid.

10.16



Volume $\sim s^3$

$E \sim \frac{Q}{s^2}$

$p = Qs$

Holding p constant, $Q \sim \frac{p}{s}$ as s changes

$$E \times \text{Volume} \sim \frac{Q}{s^2}\cdot s^3 \sim \frac{p}{s}\cdot\frac{1}{s^2}\cdot s^3 \sim p$$

Hence $\int \mathbf{E}\,dv$ over "near region" remains constant as s shrinks down, if p is kept constant.

In the case of the current loop with magnetic dipole moment m:



Volume $\sim r^3$

$B \sim \frac{I}{r}$

$m \sim I r^2$

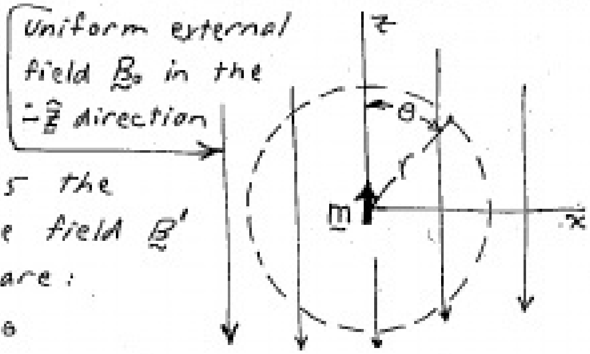
Holding m constant, $I \sim \frac{m}{r^2}$

$$B \times \text{Volume} \sim \frac{I}{r}\cdot r^3 \sim \frac{m}{r^2}\cdot\frac{1}{r}\cdot r^3 \sim m$$

Hence $\int \mathbf{B}\,dv$ over "near region" remains constant as r shrinks down, if m is kept constant.

10.17 The field of the induced dipole m is $\sim \frac{m}{r^3}$. Hence if the field is equal to the ambient field (earth's field) one boat-length L away, it will have fallen to $1/10^5$ of that magnitude at a distance of $\sqrt[3]{10^5}\,L$, or roughly $45\,L$. For a 200 ft boat this would be about 1.5 nautical miles. Working to this limit, an airplane could sweep out a searched lane about 3 miles wide. If the sensitivity were improved by a factor 2, the lane could be $\sqrt[3]{2}$ or 1.26 times as wide. This would reduce by 20% the aircraft-hours required to search a given ocean area.

10.19



From Eq. 14, p. 365 the components of the field $\mathbf{B}'$ of the dipole $\mathbf{m}$ are:

$$B'_x = \frac{3m\sin\theta\cos\theta}{r^3}$$

$$B'_z = \frac{m(3\cos^2\theta - 1)}{r^3}$$

Let us calculate the radial component of the total field, $\mathbf{B} = \mathbf{B}' + \mathbf{B}_0$:

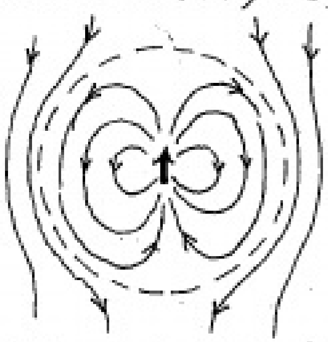
$$B'_r = B'_x\sin\theta + B'_z\cos\theta = \frac{m}{r^3}\left(3\sin^2\theta\cos\theta + 3\cos^3\theta - \cos\theta\right)$$

$$= \frac{2m}{r^3}\cos\theta$$

$$(B_0)_r = -B_0\cos\theta \qquad (\text{since } (B_0)_z = -B_0,\ (B_0)_x = 0)$$

Hence $B_r = \left(\frac{2m}{r^3} - B_0\right)\cos\theta$. On a sphere of radius $r = \left(\frac{2m}{B_0}\right)^{1/3}$, $B_r = 0$ for all values of θ.

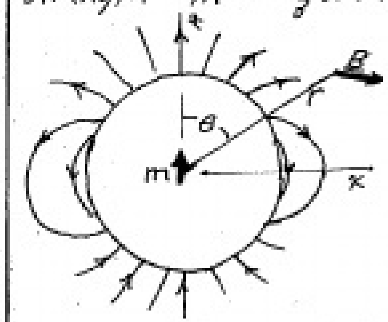
Thus no lines of force can pass through this sphere. The field outside the sphere is shown in the Figure on p. 443. Inside the sphere, it must look something like this:



If we replaced the dipole by a distribution of surface current on the sphere, so as to make the identical field outside, the field inside would be zero. For a field line inside could not end on the surface and so would have to close in some kind of loop, wholly contained inside. But in that region, it being now empty, we must have $\oint \mathbf{B}\cdot d\mathbf{s} = 0$ around any path.

10.20

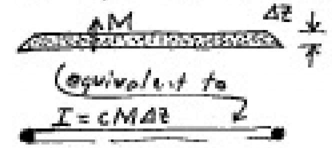
In the case of electric polarization, we learned that the external field of a uniformly polarized sphere is the same as if the total dipole moment were concentrated at the center. This must be true for the magnetized sphere also, because the external field of a magnetic dipole has the same form as that of an electric dipole. Hence the field $\mathbf{B}$ everywhere <u>outside</u> the sphere is that of a centrally located dipole of strength $m = \frac{4\pi}{3}a^3 M$.



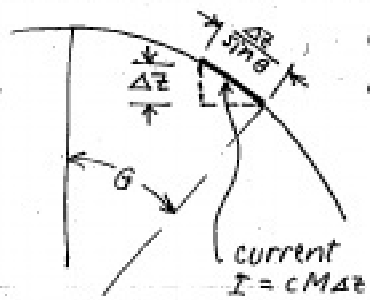
The components of $\mathbf{B}$ are

$$B_x = \frac{3m\sin\theta\cos\theta}{r^3}, \qquad B_z = \frac{m(3\cos^2\theta - 1)}{r^3} \qquad \text{outside the sphere}$$

To find the field inside, we consider the surface current to which the magnetization is equivalent. A layer of thickness Δz is equivalent to a current around the rim $I = cM\Delta z$ (as shown in Fig. 10.16). Hence if the coil shown in (c) of the Figure has its turns evenly spaced, in the z-direction, by Δz, with current $cM\Delta z$ in each turn, its field will be the same, inside and out, as that of the magnetized sphere, (except for local irregularities at the wires, if the number of turns is finite).



Letting $\Delta z \to 0$, we get a smooth surface current distribution of density $\mathcal{K} = cM\sin\theta$, since the current $I = cM\Delta z$ is spread out over a width $\frac{\Delta z}{\sin\theta}$.



Knowing the field just outside the sphere, at any position θ, and knowing the surface current density, we can

compute the field just inside the sphere, at that location. It is easier to work with $\underset{\sim}{B}$ expressed in components parallel to, and normal to, the surface of the sphere. Calling these B_r and B_θ, their relation to B_x and B_z is:

$$B_r = B_x \sin\theta + B_z \cos\theta$$
$$B_\theta = B_x \cos\theta - B_z \sin\theta$$

Outside the sphere,

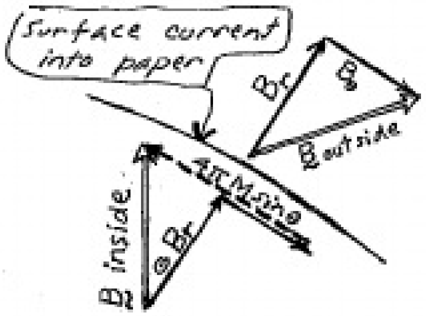
$$B_r = \frac{m}{r^3}\left(3\sin^2\theta\cos\theta + 3\cos^3\theta - \cos\theta\right) = \frac{2m}{r^3}\cos\theta$$

$$B_\theta = \frac{m}{r^3}\left(3\sin\theta\cos^2\theta - 3\sin\theta\cos^2\theta + \sin\theta\right) = \frac{m}{r^3}\sin\theta$$

at $r = a$, since $m = \frac{4\pi}{3}a^3 M$, we have, just outside the sphere:

$$B_r = \left(\frac{2}{a^3}\right)\left(\frac{4\pi}{3}a^3 M\right)\cos\theta = \frac{8\pi}{3}M\cos\theta$$
$$B_\theta = \left(\frac{1}{a^3}\right)\left(\frac{4\pi}{3}a^3 M\right)\sin\theta = \frac{4\pi}{3}M\sin\theta$$

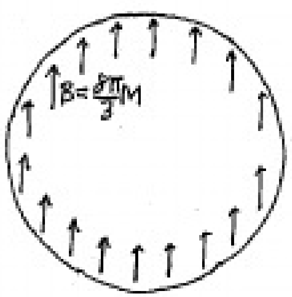
Just inside the sphere the component B_r must be the same, but the tangential component B_θ changes by $\frac{4\pi}{c}\mathcal{J} = \frac{4\pi}{c}Mc\sin\theta = 4\pi M\sin\theta$



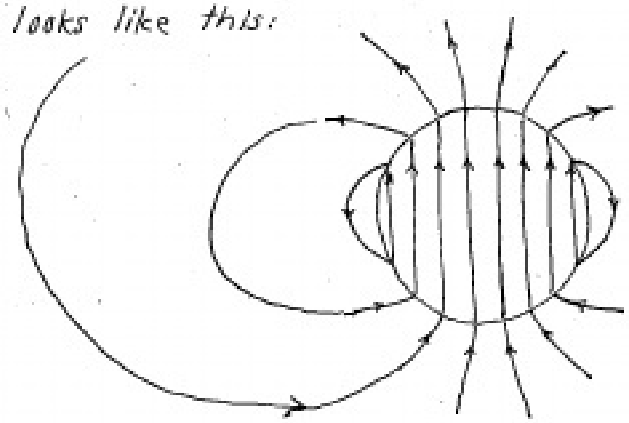
Inside: $B_r = \frac{8\pi}{3}M\cos\theta$

$$B_\theta = \frac{4\pi}{3}M\sin\theta - 4\pi M\sin\theta$$
or $$B_\theta = -\frac{8\pi}{3}M\sin\theta$$

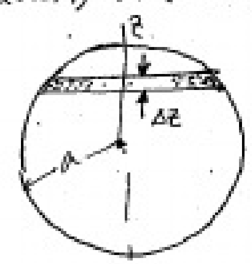
These are exactly the components of a vector in the z-direction, of magnitude $\frac{8\pi}{3}M$. Thus the field just inside the sphere, is a constant vector in the z-direction, for all θ. But there are no sources inside the sphere, so we may conclude at once that the field is uniform throughout the entire volume, $r < a$, with magnitude $B = \frac{8\pi M}{3}$. Therefore the field inside



and outside looks like this:



The rotating electrically charged sphere considered in Problem 10.6 is equivalent to a current distribution exactly like the one we have here. To see this, consider a band of vertical height Δz on the surface of the sphere. The area of the band is $2\pi a\,\Delta z$, and for constant surface charge density σ and given angular velocity ω, the equivalent current is (constant × Δz), as on the surface of our uniformly magnetized sphere. We thus obtain an interesting addendum to the result of Problem 10.6: the magnetic field of the rotating charged sphere is precisely the same, inside and out, as the field of a uniformly magnetized sphere of equal magnetic moment. In particular, since we found $m = \frac{\omega Q R^2}{3c}$ in that Problem, and in this problem,



$B_{inside} = \frac{8\pi M}{3} = \frac{2m}{a^3}$, setting $a \equiv R$ we have, for the field strength inside the rotating charged sphere:

$$B = \frac{2}{R^3}\,\frac{\omega Q R^2}{3c} = \frac{2\omega Q}{3Rc}.$$

10.21 Maxwell's Equations:

$$\text{curl}\,\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$
$$\text{curl}\,\mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{J}$$
$$\text{div}\,\mathbf{E} = 4\pi\rho$$
$$\text{div}\,\mathbf{B} = 0$$

If magnetic charge existed, the last equation would have to be replaced, as discussed in Sect. 10.2, by $\text{div}\,\mathbf{B} = 4\pi\eta$, where η stands for the density of magnetic charge. A north pole would be a positive magnetic charge, in accordance with our conventional definition of the direction of $\mathbf{B}$.

Magnetic charge in motion with velocity $\mathbf{v}$ would constitute magnetic current. Let $\mathbf{K}$ be the magnetic current density. $\mathbf{K} = \eta\mathbf{v}$, in exact analogy to $\mathbf{J} = \rho\mathbf{v}$. The conservation of magnetic charge would then be expressed by a "continuity equation."

$$\text{div}\,\mathbf{K} = -\frac{\partial \eta}{\partial t}.$$

A magnetic current will be the source of an electric field, just as an electric current is the source of a magnetic field, so we must add to the right side of the first Maxwell equation a term $\frac{4\pi}{c}\mathbf{K}$. It's not immediately obvious what sign it ought to carry, so let's write the equation $\text{curl}\,\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \pm \frac{4\pi}{c}\mathbf{K}$. Take the divergence of both sides:

$$\text{div curl}\,\mathbf{E} \equiv 0 = -\frac{1}{c}\,\text{div}\left(\frac{\partial \mathbf{B}}{\partial t}\right) \pm \frac{4\pi}{c}\,\text{div}\,\mathbf{K}$$

Replace $\text{div}\,\mathbf{K}$ by $-\frac{\partial \eta}{\partial t}$ from the continuity equation, and $\text{div}\left(\frac{\partial \mathbf{B}}{\partial t}\right)$ by $\frac{\partial}{\partial t}\,\text{div}\,\mathbf{B} = \frac{\partial}{\partial t}(4\pi\eta)$. We then have the condition $-\frac{4\pi}{c}\frac{\partial \eta}{\partial t} \pm \frac{4\pi}{c}\left(-\frac{\partial \eta}{\partial t}\right) = 0$, so the sign has to be −.

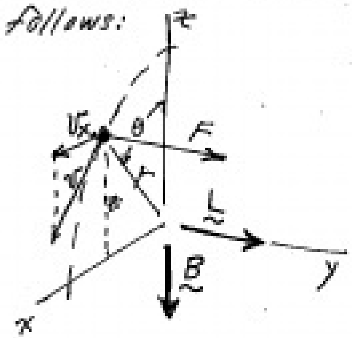
The augmented Maxwell's equations now read:

η = magnetic charge density

$\mathbf{K}$ = magnetic current density

$$\text{curl}\,\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} - \frac{4\pi}{c}\mathbf{K}$$
$$\text{curl}\,\mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{J}$$
$$\text{div}\,\mathbf{E} = 4\pi\rho$$
$$\text{div}\,\mathbf{B} = 4\pi\eta$$

10.22 The blanks are filled in as follows:

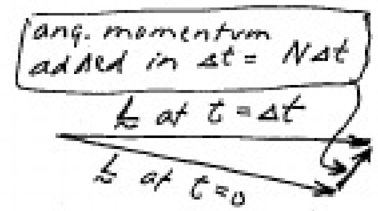


.... is a vector in the $\hat{\mathbf{y}}$ direction, of magnitude $L = Mvr$.

... a new force, of magnitude $\frac{Bqv_x}{c} = \frac{Bqv\cos\theta}{c}$ and direction $\hat{\mathbf{y}}$ to act on the particle. ...

The instantaneous torque about the $\hat{\mathbf{x}}$ axis has the magnitude $Fz = Fr\cos\theta = \frac{Bqvr}{c}\cos^2\theta$, and the average over a whole circle is $\frac{1}{2}\frac{Bqvr}{c}$. [The average of $\cos^2\theta$ is $\frac{1}{2}$.]

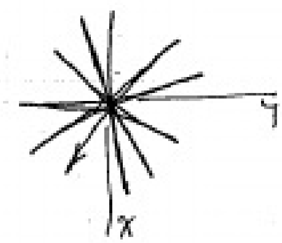
The average torque is a vector $\mathbf{N}$ of magnitude $\frac{Bqvr}{2c}$ and direction $-\hat{\mathbf{x}}$



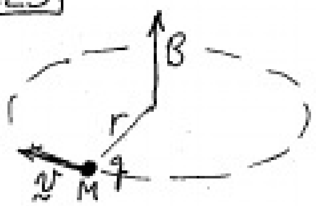
The angular momentum vector $\mathbf{L}$ will, in the course of time, precess in the horizontal plane, with a precession angular velocity ω_p given by $N/L = \frac{Bq}{2Mc}$

Comparing this with what happened when the orbit was perpendicular to the field, we observe that the rate of precession around the z axis is the same – see Eq. 33, p. 375.

As seen by an observer looking down from high up on the z-axis the trajectory of the particle in the course of many revolutions would trace out a picture like this:



10.23



The cyclotron condition is:

$$\frac{Mv^2}{r} = \frac{qvB}{c}, \text{ or}$$

$$\boxed{v = \frac{qBr}{Mc}} \qquad (1)$$

If B is changed from B to B + dB, the change in velocity, dv, is given by Eq. 28, p. 373:

$$\boxed{dv = \frac{qr}{2Mc}\,dB} \qquad (2)$$

The magnetic moment of the orbit is $\frac{qrv}{2c}$ (Eq. 21)

Solve (1) for B: $B = \frac{Mc}{q}\cdot\frac{v}{r}$

Differentiate: $dB = \frac{Mc}{q}\left(\frac{dv}{r} - \frac{v\,dr}{r^2}\right)$

Substitute into (2): $dv = \frac{r}{2}\left(\frac{dv}{r} - \frac{v\,dr}{r^2}\right)$, or

$\frac{dv}{v} + \frac{dr}{r} = 0$ Integrating: $\ln v + \ln r = \text{constant}$.

Hence rv = constant. Hence the magnetic moment $\frac{q}{2c} rv$ remains constant as B is changed.

Now differentiate (1): $dv = \frac{q}{Mc}(r\,dB + B\,dr)$

Substitute into (2): $\frac{q}{Mc}(r\,dB + B\,dr) = \frac{q}{2Mc} r\,dB$, or

$\frac{dB}{B} + \frac{2dr}{r} = 0$. Integrating: $\ln B + 2\ln r = \text{constant}$

which means that Br^2 remains constant as B is changed. The flux Φ through the orbit is $\pi r^2 B$. Hence the flux enclosed by the orbit remains constant as B is changed.

10.24 Seen in a frame of reference moving upward with velocity v_{0z}, at the instant depicted in the Figure, the particle appears to be moving in a horizontal circle with speed v_{0y}, while the field strength $|B_z|$ is increasing in magnitude. Let r_0 be the initial orbit radius and B_0 the initial field strength. According to the results of Problem 10.23, r must decrease gradually, so as to keep the flux Φ through the orbit constant, while v increases so as to keep the orbital magnetic moment constant. In the moving frame this increase in v is attributable to the E induced by the changing B.

In the lab frame, however, there is no E, and the magnitude of v must remain constant. The increase of the horizontal velocity component, call it v_h, is compensated by the decrease in the vertical component v_z as the trajectory is deflected into a flatter and flatter spiral. That is, $v_h^2 + v_z^2$ remains constant. But this can only go on until v is exactly horizontal, with $v_h^2 = v_{0y}^2 + v_{0z}^2$ and $v_z = 0$. Then the path must spiral back down. Given that $v_h r$ is a constant, the radius of the orbit at the turning-back time must be $r_0\frac{v_{0y}}{\sqrt{v_{0y}^2+v_{0z}^2}}$, and

Given that the flux $\Phi = Br^2$ enclosed by the orbit remains constant, B at the turning point must be:

$$B = B_0\frac{r_0^2}{r^2} = B_0\frac{v_{0y}^2 + v_{0z}^2}{v_{0y}^2} = B_0\left(1 + \frac{v_{0z}^2}{v_{0y}^2}\right)$$

This is the principle of the "magnetic mirror," or "magnetic bottle" used to confine ions in fusion experiments, and it is what keeps electrons in the "Van Allen belt."

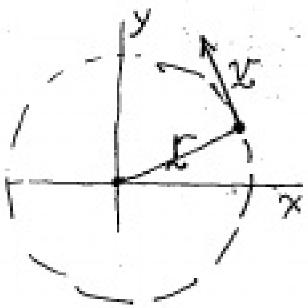
10.25 The vector potential as given, $\mathbf{A} = \frac{B}{2}(-\hat{x}y + \hat{y}x)$ corresponds to a magnetic field:

$$\mathbf{B} = \nabla \times \mathbf{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{By}{2} & +\frac{Bx}{2} & 0 \end{vmatrix} = +B\hat{z}$$

But we wanted a field in the $-\hat{z}$ direction, to agree with Fig. 10.12. So the $\mathbf{A}$ given is wrong and a correct $\mathbf{A}$ is:

$$\mathbf{A} = \frac{B}{2}(\hat{x}y - \hat{y}x)$$

Let's write out $\mathbf{r}$ and $\mathbf{v}$ as vectors:



Looking down on the orbit in Fig. 10.12 the sense of rotation is counter-clockwise.

$$\mathbf{r} = \hat{x}x + \hat{y}y$$

$$\mathbf{v} = v\left[\hat{x}\left(\frac{-y}{r}\right) + \hat{y}\left(\frac{x}{r}\right)\right]$$

With $B = 0$, $v = v_0$, the angular momentum is

$M\mathbf{r}\times\mathbf{v}_0 = Mv_0\frac{x^2+y^2}{r}\hat{z} = Mv_0 r\hat{z}$, a vector in the positive $\hat{z}$ direction, as it should be. After the field $\mathbf{B}$ is switched on, the speed has increased by $\Delta v = \frac{qrB}{2Mc}$ (Eq. 29). The change in the vector $M\mathbf{r}\times\mathbf{v}$ is $M\mathbf{r}\times\Delta\mathbf{v}$ or just $M\Delta v\, r\hat{z}$

$= \frac{qr^2B}{2c}\hat{z}$.

The vector $\mathbf{r}\times\left(\frac{q}{c}\mathbf{A}\right)$ is now $\frac{qB}{2c}(\hat{x}x+\hat{y}y)\times(\hat{x}y$

$$= \frac{qB}{2c}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & 0 \\ y & -x & 0 \end{vmatrix} = -\frac{qB}{2c}(x^2+y^2)\hat{z} = -\frac{qr^2B}{2c}\hat{z}$$

Hence the vector $r\times(M\mathbf{v} + \frac{q}{c}\mathbf{A})$ has remained cons.

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